Tentamen Continue Matrixgroepen, Voorjaar 2020/21
17 juni 2021, 08:30-11:30 (12:00)

17.06.2021

Toelichting:

- Je mag geen hulpmiddelen gebruiken behalve het boek "Lie Groups, Lie Algebras, and Representations" van Brian Hall en de teksten die ik via Brightspace verspreid heb.
- Als je stellingen uit het boek gebruikt, geef volledige referenties, waarbij je ook duidelijk maakt dat aan de voorwaarden voldaan is. Naar stellingen of huiswerkopgaven in het boek mag alleen verwezen worden als die dit semester daadwerkelijk behandeld/opgegeven zijn.
- **Schrijf op je uitwerking "Ik verklaar bij deze dat ik geen andere dan de toegestane hulpmiddelen gebruikt heb".**

**The problems are all on the next page
(so that you need to scroll less)**

1. (5pt) Compute e^X for $X = \begin{pmatrix} 4 & 3 \\ -1 & 2 \end{pmatrix}$ using Hall's Exercise 2.6.

2. (10 pt) Let $N \in \mathbb{N}$ and let V_N be the vector space of polynomials over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ of degree $\leq N$. Then $D : V_N \to V_N, P \mapsto P'$ (derivative) is a linear map. Prove that

$$(e^{tD}P)(x) = P(x+t) \qquad \forall N \in \mathbb{N},\ P \in V_N,\ t, x \in \mathbb{R}.$$

   You may do this in any way you like as long as you don't use Taylor polynomials or series.

   Remark: This result can be generalized to certain infinite dimensional function spaces, but clearly this requires more analysis (complex or functional).

3. (10 pt) Prove that every connected subgroup of $SU(2)$ is closed. Prove that the analogous claim for $SU(3)$ is false. Hint: You may use Proposition 5.24 (which I didn't prove).

4. Let G, H be connected matrix Lie groups and $\Phi : G \to H$ a homomorphism. Let $\phi : \mathfrak{g} \to \mathfrak{h}$ be the associated Lie algebra homomorphism. Prove:

   (i) (5 pt) If ϕ is surjective then Φ is surjective. (The converse is also true, but harder to prove.)

   (ii) (5 pt) ϕ is injective if and only if Φ is locally injective, i.e. there is an open neighborhood U of e_G with $\Phi_{|U}$ injective.

   (iii) (10 pt) Φ is a covering map if and only if ϕ is bijective. This doesn't require the unproven converse in (i)!

5. Consider the basis of $\mathfrak{su}(3)$ used by Hall. Let $\mathfrak{h} = \mathbb{C}H_1 + \mathbb{C}H_2 \subset \mathfrak{su}(3)$. Put

$$\begin{aligned} N &= \{A \in SU(3) \mid \mathrm{Ad}_A(H) \in \mathfrak{h}\ \forall H \in \mathfrak{h}\}, \\ Z &= \{A \in SU(3) \mid \mathrm{Ad}_A(H) = H\ \forall H \in \mathfrak{h}\}. \end{aligned}$$

   (i) (5 pt) Prove that Z is a normal subgroup of N.

Bonus (ii) (5 pt) Prove that the quotient group N/Z is finite.

Solutions

Oplossing 1 We have $\mathrm{Tr}(X) = 6$. Thus $Y = X - 3 \cdot \mathbf{1} = \begin{pmatrix} 1 & 3 \\ -1 & -1 \end{pmatrix}$ is traceless. Since $\det Y = 2$, Exercise 2.6 gives $e^Y = \cos\sqrt{2}\mathbf{1} + \frac{\sin\sqrt{2}}{\sqrt{2}}Y$. Now $e^X = e^{Y+3\mathbf{1}} = e^3(\cos\sqrt{2}\mathbf{1} + \frac{\sin\sqrt{2}}{\sqrt{2}}Y)$.

Oplossing 2 A basis for V_N is $\{1, x, x^2, \ldots, x^N\}$. Clearly it suffices to prove the claim for the basis elements, i.e. $(e^{tD}x^n)(x) = (x+t)^n$. We compute

$$\begin{aligned} e^{tD}x^n &= \sum_{k=0}^{\infty} \frac{t^k D^k}{k!} x^n = \sum_{k=0}^{n} \frac{n(n-1)\cdots(n-k+1)t^k}{k!} x^{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} t^k = (x+t)^n. \end{aligned}$$

Since this holds for all n, the claim is proven. (Of course one can write down a matrix representing D w.r.t. the given basis and then exponentiate is, but this was not required.)

Remark: The above argument is easily adapted to functions defined by a convergent power series, as long as x and $x+t$ are in the convergence disk. In particular e^{tD} is defined on the space of entire functions, for all $t \in \mathbb{R}$. In a different direction, let $V = L^2(\mathbb{R})$. Then $H = -i\frac{d}{dx}$ is a self-adjoint operator (unbounded, but densely defined), and using the functional calculus for unbounded self-adjoint operators one obtains a unitary one-parameter group $U(t) = e^{itH} = e^{tD}$. Now it is not difficult to prove $(U(t)f)(x) = f(x+t)$ for all $f \in V, t \in \mathbb{R}$ and almost all $x \in \mathbb{R}$.

Oplossing 3 Like every non-zero Lie algebra, $\mathfrak{su}(2)$ has one-dimensional Lie subalgebras. But we have seen that $\mathfrak{su}(2)$ has no two-dimensional subalgebras. The one-dimensional subalgebras clearly are abelian, and they are maximal abelian since there is no strictly larger abelian Lie subalgebra containing them. Now Proposition 5.24 (not proven in the lecture) gives that the corresponding (abelian) subgroup $H \subseteq SU(2)$ is closed. If $\alpha \in \mathbb{R}$ is irrational, the subgroup $H = \mathrm{diag}(e^{ix}, e^{i\alpha x}, e^{-ix(1+\alpha)}) \mid x \in \mathbb{R}\} \subseteq SU(3)$ is non-closed, its closure being $\overline{H} = \{\mathrm{diag}(e^{is}, e^{it}, e^{-i(s+t)}) \mid s, t \in \mathbb{R}\} \cong \mathbb{T}^2$. (Note that the top-left 2×2 corner of H is the irrational-line subgroup of the 2-torus $\mathbb{T}^2$.)

Oplossing 4 (i) Assume ϕ is surjective. Then for every $Y \in \mathfrak{h}$ there exists $X \in \mathfrak{g}$ such that $\phi(X) = Y$, thus $e^Y = e^{\phi(X)} = \Phi(e^X)$. Thus $e^Y \in \Phi(G)$ for all $Y \in \mathfrak{h}$. Since H is connected, every $B \in H$ is of the form $e^{Y_1} \cdots e^{Y_n}$. Since each of the e^{Y_i} is in $\Phi(G)$, so is B, thus Φ is surjective. (For a proof of the converse in the matrix group setting see Rossmann's book, p.80. But here the abstract manifold perspective might be better: Since we are dealing with a group homomorphism, the differential $d\Phi : T_gG \to T_{\Phi(g)}H$ has constant rank. Now the rank theorem implies that in suitable local coordinates, Φ looks like $(x_1, \ldots, x_{\dim(G)}) \mapsto (x_1, \ldots, x_r, \mathbf{0}_{\mathbb{R}^{\dim(H)-r}})$. If Φ is surjective then $r = \dim(H)$, thus Φ locally is a projection and locally admits a section Ψ, e.g. $(x_1, \ldots, x_r) \mapsto (x_1, \ldots, x_r, 0, \ldots, 0)$. Now $d\Psi : \mathfrak{h} = T_eH \to T_eG = \mathfrak{g}$ is a section for $\phi = d\Phi$, thus ϕ is surjective.)

(ii) ϕ is determined by $e^{\phi(X)} = \Phi(e^X)\ \forall X \in \mathfrak{g}$. Since $\exp : \mathfrak{g} \to G$ and $\exp : \mathfrak{h} \to H$ are local bijections (Hall's Corollary 3.44), we have $\Phi = \exp \circ \phi \circ \log$ near e_G. This implies that Φ is locally injective if and only if ϕ is locally injective. Since ϕ is linear, local injectivity is equivalent to injectivity. (If $\ker\phi \neq \{0\}$, it contains elements arbitrarily close to zero, which is ruled out by local injectivity. The other implication is trivial.)

(iii) If ϕ is bijective, it has an inverse $\psi : \mathfrak{h} \to \mathfrak{g}$. As in Hall, Section 5.7, the latter can be exponentiated locally: $\Psi(A) = e^{\psi(\log(A))}$ on some open neighborhood $V \ni e_H$. Since $\exp, \log$ are locally inverse to each other on G and H, it is clear that Ψ is a local inverse of Φ. Thus Φ restricts to a homeomorphism from some neighborhood of e_G to some neighborhood of e_H. We also need some global considerations. By (i) and (ii), bijectivity of ϕ implies that Φ is surjective and locally injective. The second fact (already known from the local invertibility just proven) implies that there is an open neighborhood V of e_G such that the closed normal subgroup $K = \ker \Phi \subseteq G$ intersects V only in e_G. By the above, we can choose $V \ni e_G$ small enough so that $\Phi_{|V}$ is a homeomorphism. We also choose V so small that $kV \cap k'V = \emptyset$ whenever $k, k' \in K, k \neq k'$. Now it is clear that Φ is a homeomorphism from each kV to $W = \Phi(V) \subset H$. By construction, we have $\Phi^{-1}(W) = \bigcup_{k \in K} kV$. Thus Φ is a covering map at the unit e_H, and elsewhere this follows from Φ being a homomorphism.

Now assume Φ is a covering map. Then it is locally injective, and (ii) gives injectivity of ϕ. Since we omitted the harder converse of (i), we must proceed differently to obtain surjectivity of ϕ: Φ being covering map, there is an open neighborhood $U \subseteq G$ of e_G such that $\Phi : U \to \Phi(U)$ is a homeomorphism. Replacing U by a smaller open set if necessary, we can achieve that U and $\Phi(U)$ are homeomorphic (via $\exp, \log$) to open neighborhoods of $0_\mathfrak{g}$ and $0_\mathfrak{h}$, respectively. By the construction of ϕ from Φ we have $\Phi(e^X) = e^{\phi(X)}$ for all $X \in \mathfrak{g}$. Since $\exp : \mathfrak{g} \to G$ and $\exp : \mathfrak{h} \to H$ are local homeomorphisms, it follows that $\phi = \log \circ \Phi \circ \exp$ is a local homeomorphism from a neighborhood of $0_\mathfrak{g}$ to a neighborhood of $0_\mathfrak{h}$. Thus $\phi(\mathfrak{g}) \subseteq \mathfrak{h}$ contains an open neighborhood of $0_\mathfrak{h}$, and together with linearity of ϕ this implies its surjectivity.

Oplossing 5 (i) It is clear enough that N is a group and $Z \subseteq N$ a subgroup. Let $z \in Z, n \in N, h \in H$. Then by definition of N we have $n^{-1}hn \in H$. Thus $(n^{-1}hn)z = z(n^{-1}hn)$ by definition of Z. Multiplying this with n from the left and with n^{-1} from the right (thus applying Ad_n) we get $hnzn^{-1} = nzn^{-1}h$. Thus nzn^{-1} commutes with h. Since $h \in H$ was arbitrary, we conclude $nzn^{-1} \in Z$, proving that $Z \subseteq N$ is normal.

(ii) is contained in the proof of Hall's Proposition 6.20 (just behind the material covered in the course).