Exam Functional Analysis, Najaar 2021/22

14 januari 2022, 08:30-11:30 (12:00)

Please note:

- You may use my lecture notes, preferably the most recent version, in printed form. Everything else (books, homework, notes, and any sort of electronic device) is prohibited.
- If you refer to a (non-trivial) result in the lecture notes, give a precise reference.
- Exercises in the lecture notes may only be cited if they were assigned this semester!
- If you can't solve a part of a problem, you may nevertheless use it for the subsequent parts of that problem!
- Write your name and student number on every sheet.
- The exam is stated in English like the lecture notes and the homework exercises, but you may of course write your solution in Dutch if you prefer.
- Do not waste time and space by giving excessively detailed proofs of easy statements!
- You can achieve 50 points (plus 5 bonus points). 28 are sufficient for passing.

Problem 1 (10 pt=2+6+2) Let H be a Hilbert space.

(i) Show briefly that $\widehat{H} = H \oplus H$ is a Hilbert space with the obvious choice of inner product.

(ii) Let $A \in B(H)$ and A^* its adjoint. Show that the graph $\mathfrak{G}(A^*) \subseteq \widehat{H}$ of A^* is the orthogonal complement in $\widehat{H}$ of a certain linear subspace.

(iii) Conclude that A^* is bounded. (Never mind that there are simpler ways of seeing this.)

Problem 2 (10 pt=5+2+3) Let E, F be Banach spaces and $A \in B(E, F)$. If A is bounded below then clearly it is injective, and we have proven that the image $AE \subseteq F$ is closed.

(i) Prove the following converse: If A is injective and $AE \subseteq F$ is closed then A is bounded below.

(ii) Give examples showing that neither of the two conditions in (i) alone implies boundedness below.

(iii) Now assume that E is a Hilbert space. Prove that $A \in B(E, F)$ has closed image $AE \subseteq F$ if and only if $A \upharpoonright (\ker A)^\perp$ is bounded below.

Continued on next side!!

Problem 3 (8pt) Let V be a Banach space and $W \subseteq V$ a closed linear subspace. Define

$$W^\perp = \{\varphi \in V^* \mid \varphi(x) = 0 \ \forall x \in W\} \subseteq V^*.$$

(i) Construct an isometric linear bijection $\alpha : (V/W)^* \to W^\perp$.

(ii) Construct an isometric linear bijection $\beta : V^*/W^\perp \to W^*$. Hint: Hahn-Banach.

Problem 4 (12 pt=2+2+2+2+2+2) Let $V = C^1([0,1], \mathbb{C})$ (the once continuously differentiable functions $[0,1] \to \mathbb{C}$). For $f \in V$ define $\|f\| = \|f\|_\infty + \|f'\|_\infty$.

(i) Prove that $(V, \|\cdot\|)$ is a Banach space. (You may assume from analysis that if $f_n, g \in V$ and $f_n \to g$ and $f'_n \to h$ uniformly then $g' = h$.)

(ii) Show that V is a Banach algebra when multiplication of functions is defined point-wise, i.e. $(fg)(x) = f(x)g(x)$.

(iii) Let $g(x) = x$ for all $x \in [0,1]$. Compute the norm, the spectrum and the spectral radius of g.

(iv) Define a $*$-operation on V by $f^*(x) = \overline{f(x)}$. Does this turn V into a C^*-algebra?

(v) If $E \subseteq [0,1]$ is a closed subset, show that

$$I_E = \{f \in V \mid f(x) = 0 \ \forall x \in E\}$$

is a closed two-sided ideal in V.

(vi) For $a \in [0,1]$ let $I_a = \{f \in V \mid f(a) = f'(a) = 0\}$. Show that I_a is a closed ideal that is not of the form I_E as in (v) for any E.

Problem 5 (10=2.5+1.5(+5)+2+2+2 pt) Let H be a Hilbert space.

(i) Prove that $\{A \in B(H) \mid A \text{ is bounded below}\}$ is an open subset of $B(H)$.

(ii) Let $A \in B(H)$ be a boundary point of the set of the set of invertible operators. Prove that A is not invertible.

(iii) [BONUS] For A as in (ii), prove that A is not bounded below.

Now for $A \in B(H)$ define the approximate point spectrum by

$$\sigma_{ap}(A) = \{\lambda \in \mathbb{C} \mid A - \lambda\mathbf{1} \text{ is not bounded below}\}.$$

(iv) Prove $\sigma_p(A) \subseteq \sigma_{ap}(A) \subseteq \sigma(A)$.

(v) Prove that $\sigma_{ap} \subseteq \mathbb{C}$ is closed.

(vi) Prove that the boundary of $\sigma(A)$ is contained in $\sigma_{ap}(A)$.

Solutions

Solution 1 (i) We give $\widehat{H}$ the inner product $\langle (x,y),(x',y')\rangle = \langle x,x'\rangle + \langle y,y'\rangle$. This clearly is an inner product, and completeness of $\widehat{H}$ w.r.t. to this inner product is straightforward (and was a homework exercise).

(ii) By definition, $\mathfrak{G}(A^*) = \{(x, A^*x) \mid x \in H\}$. Now, for $(y,z) \in \widehat{H}$ we have

$$\langle (x,A^*x),(y,z)\rangle = \langle x,y\rangle + \langle A^*x,z\rangle = \langle x,y\rangle + \langle x,Az\rangle = \langle x, y+Az\rangle.$$

We have $(y,z) \in \mathfrak{G}(A^*)^\perp$ if and only this vanishes for all $x \in H$, which is equivalent to $y+Az = 0$ and thus to $y = -Az$. Thus $\mathfrak{G}(A^*)^\perp = \{(-Az, z) \mid z \in H\} =: V$. The orthogonal complement of any subset of a Hilbert space is a closed linear subspace, thus V is a closed linear subspace. (Note that $V = \mathfrak{G}(-A)^t$, where $(x,y)^t = (y,x)$.) If we prove that $\mathfrak{G}(A^*) = V^\perp$, we are done. From $\mathfrak{G}(A^*)^\perp = V$ we obtain $\mathfrak{G}(A^*)^{\perp\perp} = V^\perp$, but to replace the l.h.s. by $G(A^*)$ we'd need to know that space to be closed, which is what we want to prove! Thus we have to compute $V^\perp$ directly: If $(x,y) \in \widehat{H}$ then

$$\langle (x,y),(-Az,z)\rangle = \langle x,-Az\rangle + \langle y,z\rangle = -\langle A^*x,z\rangle + \langle y,z\rangle = \langle y - A^*x, z\rangle.$$

Now $(x,y) \in V^\perp$ is equivalent to the above vanishing for all $z \in H$, thus to $y = A^*x$ and therefore to $(x,y) \in \mathfrak{G}(A^*)$. Thus indeed $\mathfrak{G}(A^*) = V^\perp$.

Remark: Strictly speaking, one can simply write down the definition of V and prove $G(A^*) = V^\perp$, but that is bad style. Where does that V come from?

(iii) Since orthogonal complements in a Hilbert space are always closed, it follows from (ii) that the graph of A^* is closed. Now boundedness of A follows from the closed graph theorem. ■

Solution 2 (i) By assumption, the image $F' = AE \subseteq F$ is closed, thus a Banach space. The map $A' : E \to F'$ given by interpreting A as a map $E \to F'$ is injective and surjective. Thus by the bounded inverse theorem there is a bounded linear map $(A')^{-1} : F' \to E$ inverting A'. I.e. there is a C such that $\|x\| = \|(A')^{-1}Ax\| \le C\|Ax\|$ for all $x \in E$. Equivalently, $\|Ax\| \ge C^{-1}\|x\|\ \forall x \in E$, so that A is bounded below.

(ii) Let $E = F = \ell^2(\mathbb{N},\mathbb{C})$ and $A \in B(E,F)$ defined by $(Af)(n) = f(n)/n$. Then $Af = 0$ if and only if $f = 0$, thus A is injective. But it is not bounded below. And any zero operator has closed image $\{0\}$, but is not bounded below.

(iii) Combining (i) and Lemma 7.33 we have: An injective bounded linear map $A \in B(E,F)$ has closed image if and only if it is bounded below. Taking into account that $AE = A(\ker A)^\perp$ and that the restriction $A \upharpoonright (\ker A)^\perp$ is injective, the conclusion is immediate. ■

Solution 3 (i) Let $p : V \to V/W$ be the quotient map. If $\psi \in (V/W)^*$ then clearly $\psi \circ p$ is in V^* and vanishes on W, thus $\psi \circ p \in W^\perp$. This defines a map $\alpha : (V/W)^* \to W^\perp$. If $\psi \ne \psi'$ then surjectivity of p implies $\psi \circ p \ne \psi' \circ p$, thus α is injective. If $\varphi \in W^\perp$, thus $\varphi \upharpoonright W = 0$, then Proposition 6.1(v) implies that there is a unique $\psi : V/W \to \mathbb{F}$ such that $\psi \circ p = \varphi$. This proves that α is surjective. That α is isometric also follows from Proposition 6.1(v).

(ii) Let $p : V^* \to V^*/W^\perp$ be the quotient map. If $\varphi \in V^*$ then the restriction $\varphi \upharpoonright W$ clearly is in W^*. And $\varphi \in W^\perp$ implies $\varphi \upharpoonright W$ is zero. Thus there is a well-defined linear map $\beta : V^*/W^\perp \to W^*$. If $\beta(\varphi) = \varphi \upharpoonright W = 0$ then $\varphi \in W^\perp$, so that β is injective. Since $\|\varphi \upharpoonright W\| \le \|\varphi\|$, β has norm ≤ 1. By the Hahn-Banach theorem, for every $\psi \in W^*$ there is a $\widehat{\psi} \in V^*$ of the same norm such that $\widehat{\psi} \upharpoonright W = \psi$. Now $\beta(p(\widehat{\psi})) = \psi$, so that β is surjective and has norm ≥ 1. In particular it is isometric. ■

Solution 4 (i) It is quite obvious that V is a vector space and $\|\cdot\|$ a norm on it. Nevertheless, I expect to see at least a few comments on this! It remains to show that $(V, \|\cdot\|)$ is complete. If $\{f_n\} \subseteq V$ is a Cauchy sequence w.r.t. $\|\cdot\|$ then $\|f\|_\infty \le \|f\|$ implies that $\{f_n\}$ is Cauchy w.r.t. $\|\cdot\|_\infty$ and therefore uniformly converges to some $g \in C([0,1],\mathbb{C})$. And $\{f'_n\}$ is Cauchy w.r.t. $\|\cdot\|_\infty$ and converges uniformly to some $h \in C([0,1],\mathbb{C})$. Now the stated fact implies $g \in V$. Thus $\|f_n - g\|_\infty \to 0$ and $\|f'_n - g'\|_\infty \to 0$, implying also $\|f_n - g\| \to 0$. Thus V is complete.

(ii) If $f, g \in V$ then clearly $fg \in V$. I am shocked that more than one person claimed $(fg)' = f'g'$! Now submultiplicativity of $\|\cdot\|$ follows from the computation

$$\begin{aligned}
\|fg\| &= \|fg\|_\infty + \|(fg)'\|_\infty \le \|f\|_\infty\|g\|_\infty + \|f'g + fg'\|_\infty \\
&\le \|f\|_\infty\|g\|_\infty + \|f\|_\infty\|g'\|_\infty + \|f'\|_\infty\|g\|_\infty \\
&\le \|f\|_\infty\|g\|_\infty + \|f\|_\infty\|g'\|_\infty + \|f'\|_\infty\|g\|_\infty + \|f'\|_\infty\|g'\|_\infty \\
&= (\|f\|_\infty + \|f'\|_\infty)(\|g\|_\infty + \|g'\|_\infty) = \|f\|\|g\|.
\end{aligned}$$

(iii) Easily, $\|g\| = \|x\|_\infty + \|1\|_\infty = 2$. If $\lambda \in [0,1]$ then $g - \lambda\mathbf{1}$ is not invertible, while $x \mapsto (x-\lambda)^{-1}$ is a smooth function, thus in V, if $\lambda \notin [0,1]$. Thus $\sigma(g) = [0,1]$ and $r(g) = 1$.

(iv) It is easy to see that $*$ is an involution/$*$-operation. But it does not turn V into a C^*-algebra! This follows either by directly showing that the C^*-identity fails, of from (iii) since in a commutative C^*-algebra we would have $\|g\| = r(g)\ \forall g$, whereas (iii) provides a counterexample.

(v) I don't feel like writing up the rather obvious proof.

(vi) Clearly I_a is a linear subspace of V. If $f \in I_a$, $g \in V$ then $(fg)(a) = 0$ and $(fg)'(a) = f(a)g'(a) + f'(a)g(a) = 0$, thus $fg \in I_a$. Thus I_a is an ideal, and closedness w.r.t. $\|\cdot\|$ is clear in view of the definition of the norm. Clearly I_a is contained in $I_{\{a\}}$, but it is strictly smaller since the function $x \mapsto x-a$ is in $I_{\{a\}}$ but not in I_a. On the other hand, $f : x \mapsto (x-a)^2$ is in I_a, but not in I_E for any $E \supsetneq \{a\}$. Thus I_a is not of the form I_E for any closed $E \subseteq [0,1]$. ■

Solution 5 (i) Let $A \in B(V)$ be bounded below. I.e. there exists $C > 0$ such that $\|Ax\| \ge C\|x\|\ \forall x \in V$. If $A' \in B(V)$ with $\|A - A'\| < C/2$ then using $\|y - y'\| \ge \|y\| - \|y'\|$ we have

$$\|A'x\| = \|(A - (A - A'))x\| \ge \|Ax\| - \|(A - A')x\| \ge C\|x\| - \frac{C}{2}\|x\| = \frac{C}{2}\|x\| \quad \forall x \in V,$$

so that A' is bounded below. Thus the set of bounded-below operators is open.

(ii) Let $I = \mathrm{Inv}(B(H))$. By definition of the boundary, $\partial I = \overline{I} \cap \overline{B(H)\backslash I}$. By Lemma 11.10(ii), I is open, thus $\partial I = \overline{I} \cap (B(H)\backslash I)$. Thus every $A \in \partial I$ is in $B(H)\backslash I$ and therefore not invertible.

(iv) If $\lambda \in \sigma_p(A)$ then $A - \lambda\mathbf{1}$ is not injective and therefore not bounded below. Thus $\lambda \in \sigma_{ap}(A)$. And if $\lambda \in \sigma_{ap}(A)$, thus $A - \lambda\mathbf{1}$ is not bounded below, then it is not invertible by Proposition 7.35. Thus $\lambda \in \sigma(A)$.

(v) If $\lambda \notin \sigma_{ap}(A)$ then by (i) also $\lambda' \notin \sigma_{ap}(A)$ for $|\lambda - \lambda'|$ small enough. Thus $\mathbb{C}\backslash\sigma_{ap}(A)$ is open.

(vi) If $\lambda \in \partial\sigma(A)$ then $A - \lambda\mathbf{1} \in \partial I$ (since there are λ' arbitrarily close to λ with $A - \lambda\mathbf{1} \in$ invertible or not). Thus $A - \lambda\mathbf{1}$ is not bounded below by (iii), so that $\lambda \in \sigma_{ap}(A)$.

(iii) Claim: $\|B^{-1}\| \to \infty$ as $B \to A$ in I. Equivalently, for every $M > 0$ there is an open neighborhood $U \subseteq B(V)$ of A such that $\|B^{-1}\| > M$ for all $B \in I \cap U$.

Proof of the claim: Falsity of the claim would mean that there is an $M > 0$ such that for every neighborhood U of A there is a $B \in U \cap I$ with $\|B^{-1}\| \le M$. In particular we can find $B \in I$ with $\|B^{-1}\| \le M$ and $\|A - B\| < M^{-1}$. This implies $\|A - B\| < \|B^{-1}\|^{-1}$, which together with invertibility of B implies invertibility of A by the proof of Lemma 11.10(ii), contradicting (ii). ■

Proof of (iii): Let $\varepsilon > 0$. By the claim, we can find $B \in I$ such that $\|A - B\| < \varepsilon/2$ and $\|B^{-1}\| > 3/\varepsilon$. By definition of the norm, we can find $x \in V$ with $\|x\| = 1$ and $\|B^{-1}x\| > 2/\varepsilon$. Put $y = \frac{B^{-1}x}{\|B^{-1}x\|}$. Now $Ay = (A - B)y + By = (A - B)y + \frac{x}{\|B^{-1}x\|}$. With $\|x\| = \|y\| = 1$ we have $\|Ay\| \leq \|A - B\| + \frac{1}{\|B^{-1}x\|} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Since $\|y\| = 1$ and $\varepsilon > 0$ was arbitrary, we have proven that A is not bounded below. ■