Retake Functional Analysis, Najaar 2020/21

26 maart 2021, 08:30-11:30 (12:00)

Please note:

- You may use my lecture notes, notes that you have made yourself, as well as the homework excercises including feedback on them, but no books. All such materials must be in printed form since you are not allowed to access a computer, tablet or phone – except of course for participating in the zoom meeting. During the exam, you must keep your camera on, refrain from using a fake background and keep the microphone off (except for questions).
- **Place the camera far enough from you so that I can see your face, hands and what you have on the table.**
- If you refer to a (non-trivial) result in the lecture notes, give a precise reference.
- If you can't solve a part of a problem, you may nevertheless use it for the subsequent parts of that problem!
- Write your name, student number and university on the first sheet. Send me ONE PDF of your solution INCLUDING the printout or handwritten copy of the exam. If you need to go to the bathroom, ask first, preferably without turning the microphone on.
- The total number of obtainable points is 50, and 28 are certainly sufficient for passing.

Problem 1 (10 pt=3+3+4) Let H be a Hilbert space, $A \in B(H)$ and $C > 0$ such that $|\langle Ax, x\rangle| \geq C\|x\|^2 \ \forall x \in H$.

(i) Prove that A is bounded below.

(ii) For $x \in (AH)^\perp$, prove $x = 0$. Conclude that $AH \subseteq H$ is dense.

(iii) Prove that A is invertible and $\|A^{-1}\| \leq C^{-1}$.

Problem 2 (10 pt) Let X, Y, Z be Banach spaces and let $T : X \times Y \to Z$ be a map such that $X \to Z, x \mapsto T(x, y)$ is linear and bounded for each fixed $y \in Y$ and analogously for $Y \to Z, y \mapsto T(x, y)$ with fixed $x \in X$.

Prove that there is a $0 \leq C < \infty$ such that $\|T(x, y)\| \leq C\|x\|\|y\|$ for all $x \in X,\ y \in Y$.

Hint: Use the uniform boundedness theorem.

Continued on the other side!!

Problem 3 (10 pt=8+2) Let H be a Hilbert space and $T = T^* \in B(H)$ diagonalizable. Thus there is an orthonormal basis $\{e_i\}_{i\in I}$ for H consisting of eigenvectors of T with corresponding eigenvalues $\{\lambda_i\}_{i\in I}$. Let $y \in H$.

(i) Prove that there is a solution $x \in H$ of the equation $Tx = y$ if and only if

$$y \in (\ker T)^\perp \qquad \text{and} \qquad \sum_{i\in I, \lambda_i \neq 0} \frac{1}{\lambda_i^2} |\langle y, e_i\rangle|^2 < \infty. \tag{1}$$

(ii) Assume the given data satisfy (1). How can one find all solutions x of $Tx = y$?

Problem 4 (10 pt=1+3+3+3) We write $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

(i) Let H be a Hilbert space and $A \in B(H)$. Define $p_n = \|A^n\|$ for $n \in \mathbb{N}_0$ and prove that $p_{i+j} \leq p_i p_j \ \forall i, j$.

(ii) Let $\{\alpha_k\}_{k\in\mathbb{N}_0} \subseteq \mathbb{C}$. Put $H = \ell^2(\mathbb{N}_0, \mathbb{C})$ with natural ONB $\{e_k\}$ and define a linear 'weighted shift operator' $A : H \to H$ by $Ae_k = \alpha_k e_{k+1}$. Prove that $\|A\| = \sup_k |\alpha_k|$.

(iii) Let $\{p_k\}_{k\in\mathbb{N}_0}$ be given so that $p_k > 0 \ \forall k$ and $p_{i+j} \leq p_i p_j \ \forall i, j$. Define

$$\alpha_0 = p_0 \qquad \text{and} \qquad \alpha_k = \frac{p_k}{p_{k-1}} \ \text{ if } \ k \geq 1.$$

Prove that the sequence $\{\alpha_k\}$ and therefore the associated weighted shift operator A are bounded.

(iv) Let A be constructed from $\{p_k\}_{k\in\mathbb{N}_0}$ as in (iii). Prove $\|A^n\| = p_n \ \forall n \in \mathbb{N}_0$. (Hint: $\|A^n\| \leq p_n \leq \|A^n\|$.) What do you conclude from (i) en (iv)?

Problem 5 (10 pt=6+4) Let $\mathcal{A}$ be a unital C^*-algebra.

(i) For $a \in \mathcal{A}$ normal prove:

(a) $a = a^*$ if and only if $\sigma(a) \subseteq \mathbb{R}$.

(b) a is unitary if and only if $\sigma(a) \subseteq S^1$.

(c) $a^2 = a$ if and only if $\sigma(a) \subseteq \{0, 1\}$.

(ii) Since we have proven that $a^*a \geq 0 \ \forall a \in \mathcal{A}$, we can define $|a| = (a^*a)^{1/2}$. If $\mathcal{A} = B(H)$, we can always find a v such that $a = v|a|$ (polar decomposition). Give an example of a unital C^*-algebra $\mathcal{A}$ and $a \in \mathcal{A}$ such that <u>no</u> $b \in \mathcal{A}$ exists with $a = b|a|$. Hint: Try $\mathcal{A}$ commutative.

Solutions

Solution 1 (i) With Cauchy-Schwarz we have $\|Ax\|\|x\| \geq |\langle Ax, x\rangle| \geq C\|x\|^2$. For $x \neq 0$, division by $\|x\| \neq 0$ gives $\|Ax\| \geq C\|x\|$, which is also true for $x = 0$. Thus A is bounded below, thus injective and has closed image by Lemma 9.26.

(ii) If $x \in (AH)^\perp$ then $\langle Ax, x\rangle = 0$ so that our assumption implies $x = 0$. Thus $(AH)^\perp = \{0\}$, which is equivalent to $\overline{AH} = H$.

(iii) Since AH is closed and dense, we have $AH = H$, thus A is surjective. Since it also is bounded below, it has a bounded inverse by Proposition 9.28. Now Lemma 9.25 gives $\|A^{-1}\| = (\inf_{\|x\|=1} \|Ax\|)^{-1} \leq C^{-1}$.

Solution 2 For each $x \in X\backslash\{0\}$ define $A_x : Y \to Z,\ y \mapsto \frac{T(x,y)}{\|x\|}$. Since $T(x, \bullet) : Y \to Z$ is bounded for each $x \in X\backslash\{0\}$, the same holds for A_x. Thus $\mathcal{F} = \{A_x \mid x \in X\backslash\{0\}\} \subseteq B(Y, Z)$. For $y \in Y$ we have $\mathcal{F}y = \left\{\frac{T(x,y)}{\|x\|} \mid x \in X\backslash\{0\}\right\}$. The assumption that $T(\bullet, y)$ be bounded for each $y \in Y$ implies that $\mathcal{F}y \subseteq Z$ is a bounded set. Thus the family $\mathcal{F}$ is pointwise bounded and therefore, by the uniform boundedness theorem, uniformly bounded. This means that there is a $0 \leq C < \infty$ such that $\|A_x\| \leq C\ \forall x \in X\backslash\{0\}$, thus $\|A_x(y)\| \leq C\|y\|\ \forall x \in X\backslash\{0\},\ y \in Y$. Taking into account the definition of A_x this translates to $\|T(x, y)\| \leq C\|x\|\|y\|\ \forall x \in X\backslash\{0\},\ y \in Y$. Since this inequality obviously also holds for $x = 0$, we are done.

Remark: In view of the bilinearity of T, we have $A_x(y) = \frac{T(x,y)}{\|x\|} = T(\frac{x}{\|x\|}, y)$, so that $\mathcal{F} = \{T_x \mid x \in X,\ \|x\| = 1\}$, and we can apply the uniform boundedness theorem to this definition of $\mathcal{F}$.

BUT: If we put $\mathcal{F} = \{T(x, \bullet) \mid x \in X\} \subseteq B(Y, Z)$, then for $y \in Y$ we have $\mathcal{F}y = \{T(x, y) \mid x \in X\} \subseteq Z$, which clearly is unbounded (unless $T(x, y) = 0\ \forall x$). Thus $\mathcal{F}$ is **not** pointwise bounded and the UBT does not apply. Unfortunately, this mistake is very popular and it leads to a maximum of 5 points.

Here, as always, in trying to apply a theorem one must verify that its hypotheses are satisfied rather than be guided by wishful thinking!

Solution 3 (i) Assume that $Tx = y$ has a solution $x \in H$. Then $y \in TH \subseteq \overline{TH} = (\ker T)^\perp$, where we have combined the general fact $\ker T^* = (TH)^\perp$ (Lemma 13.1), equivalent to $(\ker T^*)^\perp = \overline{TH}$, with the assumption $T = T^*$.

Since $\{e_i\}$ diagonalizes T we have $T = \sum_{i\in I} \lambda_i P_i$, where $P_i : f \mapsto \langle f, e_i\rangle e_i$. Expanding $x = \sum_i c_i e_i$ where $c_i = \langle x, e_i\rangle$ and $y = \sum_i \langle y, e_i\rangle e_i$, the equation $Tx = y$ becomes

$$\sum_i c_i \lambda_i e_i = Tx = y = \sum_i \langle y, e_i\rangle e_i.$$

In view of the orthogonality of the e_i, this implies $c_i\lambda_i = \langle y, e_i\rangle$ for all i. This implies

$$\sum_{\substack{i \\ \lambda_i \neq 0}} \left|\frac{\langle y, e_i\rangle}{\lambda_i}\right|^2 = \sum_{\substack{i \\ \lambda_i \neq 0}} |c_i|^2 \leq \sum_i |c_i|^2 = \|x\|^2 < \infty,$$

which is the second condition on y. Now assume that y satisfies both conditions. Define $x = \sum_i c_i e_i$, where

$$c_i = \begin{cases} 0 & \text{if } \lambda_i = 0 \\ \frac{\langle y, e_i\rangle}{\lambda_i} & \text{if } \lambda_i \neq 0 \end{cases}$$

This $x \in H$ is well-defined by Theorem 5.41 (Riesz-Fischer) and $\sum_i |c_i|^2 < \infty$. And in view of $c_i\lambda_i = \langle y, e_i\rangle\ \forall i$, we have $Tx = y$.

(ii) If $Tx = y$ and $Tx' = y$ then $T(x - x') = 0$, thus $x - x' \in \ker T$. And if $Tx = y$ and $z \in \ker T$ then $x' = x + z$ satisfies $Tx' = y$. Thus (as in linear algebra), if we use (i) to to find a particular solution x_0 then the set of x satisfying $Tx = y$ is given by $x_0 + \ker T$.

Solution 4 (i) $p_{i+j} = \|A^{i+j}\| = \|A^i A^j\| \le \|A^i\|\|A^j\| = p_i p_j$, where we used $\|AB\| \le \|A\|\|B\|$.

(ii) Defining $B : H \to H$ by $Be_k = \alpha_k e_k$, we have $\|B\| = \sup_k |\alpha_k|$ (Exercise 14.11(ii)). With the right shift R we have $RBe_k = R\alpha_k e_k = \alpha_k e_{k+1}\ \forall k$, thus $A = RB$. Since R is an isometry, we have $\|Ax\| = \|RBx\| = \|Bx\|\ \forall x \in H$. Thus $\|A\| = \|B\| = \sup_k |\alpha_k|$.

(iii) By assumption $p_{i+j} \le p_i p_j\ \forall i, j$. For $k \ge 1$ this implies with $i = 1, p = k - 1$ that $p_k \le p_1 p_{k-1}$, to wit $p_k/p_{k-1} \le p_1$. For all $k \ge 1$ it follows that $\alpha_k := p_k/p_{k-1} \le p_1$, so that $\{\alpha_k\}$ and thus A are bounded with $\|A\| \le p_1$.

(iv) In view of $A^n e_k = \alpha_k \alpha_{k+1} \cdots \alpha_{k+n-1} e_{k+n}$ we see that A^n is a weighted shift operator and $\|A^n\| = \sup_k |\alpha_k \alpha_{k+1} \cdots \alpha_{k+n-1}|$. With our choice of $\{\alpha_k\}$ we find

$$\begin{aligned}\|A^n\| &= \sup_k |\alpha_k \alpha_{k+1} \cdots \alpha_{k+n-1}| = \sup_k \left| \frac{p_k}{p_{k-1}} \frac{p_{k+1}}{p_k} \cdots \frac{p_{k+n-1}}{p_{k+n-2}} \right| = \sup_k \left| \frac{p_{k+n-1}}{p_{k-1}} \right| \\ &\le p_n = \alpha_0 \alpha_1 \cdots \alpha_n \le \|A^n\|,\end{aligned}$$

where we have used that $\frac{p_{k+n-1}}{p_{k-1}} \le p_n$ by assumption. This entails $p_n = \|A^n\|\ \forall n$, as claimed.

The conclusion is that at least all sequences $\{p_k\}$ satisfying $p_k > 0$ and $p_{i+j} \le p_i p_j$ come from weighted shift operator A via $\|A^n\| = p_n\ \forall n$. (Allowing that $p_k = 0$ for some k is not difficult, but was not required: The hypothesis $p_{i+j} \le p_i p_j$ then implies $p_n = 0\ \forall n \ge k$, so that an associated operator A must be nilpotent. This can be achieved by replacing $H = \ell^2(\mathbb{N}_0, \mathbb{C})$ by $H = \mathbb{C}^k$.)

Solution 5 (i) The implications $a = a^* \Rightarrow \sigma(a) \subseteq \mathbb{R}$ and a unitary $\Rightarrow \sigma(a) \subseteq S^1$ are contained in Propositie 11.24. As to the converse, the C^*-subalgebra $\mathcal{B} = C^*(\mathbf{1}, a) \subseteq \mathcal{A}$ is commutative and unital, and by Theorem 12.18 we have an isometric $*$-isomorphism $\pi : C(\sigma(a), \mathbb{C}) \to \mathcal{B}$. Let $f = \pi^{-1}(a)$. Then f is the inclusion map $\sigma(a) \hookrightarrow \mathbb{C}$, thus its image is $\sigma(a)$. Thus if $\sigma(a) \subseteq \mathbb{R}$ then f is real-valued, so that $f = \overline{f} = f^*$. Now also $a = \pi(f)$ is self-adjoint. Analogously, $\sigma(a) \subseteq S^1$ implies $ff^* = f^*f = 1$, so that f is unitary, thus also a.

Finally, if $a^2 = a$ then the function $f : \sigma(a) \hookrightarrow \mathbb{C}$, for which we have $\pi(f) = a$, satisfies $f^2 = f$, thus $x^2 = x$ for all $x \in \sigma(a)$, which implies $\sigma(a) \subseteq \{0, 1\}$. This argument also works the other way round.

(ii) It is quite natural to first try a commutative C^*-algebra $\mathcal{A} = C(X, \mathbb{C})$, where X is compact Hausdorff. Each $f \in \mathcal{A}$ thus is a function $X \to \mathbb{C}$, and $f^*f = |f|^2$, thus $|f|$ is the function $x \mapsto |f(x)|$. If $f = b|f|$ then b thus is a continuous function $X \to \mathbb{C}$ such that $f(x) = b(x)|f(x)|\ \forall x \in X$. It follows that $b(x) = f(x)/|f(x)|$ whenever $f(x) \neq 0$. The problem now is to find a space X and a function $f \in C(X, \mathbb{C})$ such that the function $f_0 : x \mapsto f(x)/|f(x)|$, a priori defined on $X_0 = \{x \in X \mid f(x) \neq 0\}$, admits no continuous extension to all of X. This is not too difficult: Let $X \subseteq \mathbb{C}$ be compact with 0 as interior point and $f : X \hookrightarrow \mathbb{C}$ (the inclusion map). Now $f(x)/|f(x)| = x/|x|$ is defined on $X \backslash \{0\}$, but does not admit a continuous extension to X (since $x/|x|$ assumes all values in S^1 on any neighborhood of $x = 0$, however small).