Tentamen Functionaalanalyse, Najaar 2020/21

8 januari 2021, 08:30-11:30 (12:00)

Please note:

- You may use my lecture notes, notes that you have made yourself, as well as the homework excercises including feedback on them, but no books. All such materials must be in printed form since you are not allowed to access a computer, tablet or phone – except of course for participating in the zoom meeting. During the exam, you must keep your camera on, refrain from using a fake background and keep the microphone off (except for questions).
- If you refer to a (non-trivial) result in the lecture notes, give a precise reference.
- If you can't solve a part of a problem, you may nevertheless use it for the subsequent parts of that problem!
- Write your name, student number and university on the first sheet. Send me ONE PDF of your solution INCLUDING the printout or handwritten copy of the exam. If you need to go to the bathroom, ask first, preferably without turning the microphone on.

Problem 1 (10 pt=4+3+3 (+4)) Let H be a Hilbert space and $K \subseteq H$ a linear subspace.

(i) Prove, without invoking Hahn-Banach, that for every $\varphi \in K^*$ there exists $\widehat{\varphi} \in H^*$ such that $\widehat{\varphi} \upharpoonright K = \varphi$.

(ii) Prove that $\widehat{\varphi} \in H^*$ with $\widehat{\varphi} \upharpoonright K = \varphi$ is unique if and only $\overline{K} = H$.

(iii) Prove that there is a unique $\widehat{\varphi} \in H^*$ satisfying $\widehat{\varphi} \upharpoonright K = \varphi$ and $\|\widehat{\varphi}\| = \|\varphi\|$.

(iv) BONUS: If V is a Banach space, $K \subseteq V$ a linear subspace and $\varphi \in K^*$, we know from Hahn-Banach that there exists $\widehat{\varphi} \in V^*$ with $\widehat{\varphi} \upharpoonright K = \varphi$ and $\|\widehat{\varphi}\| = \|\varphi\|$. Give an example for a Banach space V not enjoying the uniqueness property in (iii).

Problem 2 (10 pt=4+1+3+2) Let V be a Banach space. Prove:

(i) If V^* is separable then V is separable.

(ii) If V is separable and reflexive then V^* is separable.

(iii) If S is infinite then $\ell^\infty(S, \mathbb{F})$ is not separable. Hint: $\{0,1\}^S$.

(iv) Give examples of infinite dimensional separable spaces V with (a) V^* separable and (b) V^* non-separable.

Continued on next side!!

Problem 3 (10 pt=3+4+3) (i) Let V be a Banach space and $W, Z \subseteq V$ closed linear subspaces with Z finite dimensional. Prove that $W + Z \subseteq V$ is closed. Hint: V/W

(ii) Let V, W be Banach spaces and $A \in B(V, W)$ injective such that $\dim(W/AV) < \infty$. Prove that $AV \subseteq W$ is closed. Hint: Open mapping theorem or related result.

(iii) Remove the injectivity assumption in (ii).

Problem 4 (5 pt) Prove: If $\mathcal{A}$ is a unital C^*-algebra and $c \in \mathcal{A}$ is normal then c^*c is positive. Hint: $c = a + ib$. [Of course you may not refer to or copy the proof given without normality assumption in the lecture notes. Give a simple direct argument.]

Problem 5 (15 pt= 3+3+3+3+3) Let $H = \ell^2(\mathbb{N}, \mathbb{C})$ with its canonical ONB $\{\delta_n\}$. Given a sequence $\{\lambda_k\} \subseteq \mathbb{C}$, define a linear operator $A_\lambda : H \to H$ by $A_\lambda \delta_k = \lambda_k \delta_{k+1}$ $\forall k$.

(i) State and prove conditions on $\{\lambda_k\}$ that are equivalent to A_λ being (a) bounded, (b) injective, (c) nilpotent.

(ii) For $\lambda_k = 2^{-k}$ $\forall k \in \mathbb{N}$, prove that A_λ is injective and quasi-nilpotent, but not nilpotent.

(iii) For general $\{\lambda_k\}$, compute $(A_\lambda)^*$, $|A_\lambda|$ and the partial isometry V in the polar decomposition.

(iv) Give the necessary and sufficient condition on $\{\lambda_k\}$ for $|A_\lambda|$ to be compact, and do the same for A_λ.

(v) Under which condition on $\{\lambda_k\}$ is A_λ normal? Which spectral theorems do apply to A_λ?

Solutions

Solution 1 (i) If $K \subseteq H$ is not closed, Lemma 4.17 gives a unique extension of φ to a bounded linear functional on the closure $\overline{K}$. We may thus assume K to be closed, thus a Hilbert space. By Theorem 6.29, applied to K, there is a unique $y \in K$ such that $\varphi(x) = \langle x, y\rangle$ for all $x \in K$. Defining $\widehat{\varphi}(x) = \langle x, y\rangle$ for all $x \in H$, it is clear that $\widehat{\varphi} \upharpoonright K = \varphi$.

(ii) By Theorem 6.24 we have $H \cong K \oplus K^\perp$. We suppress the isomorphism and identify H with $K \oplus K^\perp$. Every $\widehat{\varphi} \in H^*$ is of the form $\langle \bullet, (v, w)\rangle$, where $v \in K, w \in K^\perp$. It is clear that $\widehat{\varphi}$ coincides with φ on K if and only if v equals the y in (i). Thus we have uniqueness only if $K^\perp = 0$, i.e. $K = H$.

(iii) A vector $(v, w) \in K \oplus K^\perp$ has norm $\sqrt{\|v\|^2 + \|w\|^2}$, and the same holds for the $\widehat{\varphi} \in H^*$ induced by (v, w). With $v = y$, this norm coincides with $\|\varphi\| = \|y\|$ only if $w = 0$. This proves the uniqueness.

(iv, BONUS) Let $V = \ell^1(\mathbb{N}, \mathbb{C})$ and $K = \{f \in V \mid f(1) = 0\}$. Let $\varphi \in K^*\backslash\{0\}$. Then by Theorem 5.16 there is a unique function $g \in \ell^\infty(\mathbb{N}\backslash\{1\}, \mathbb{C})$ such that $\varphi(f) = \sum_{k=2}^\infty f(k)g(k)$ for all $f \in K$. Extend g to $\widehat{g} \in \ell^\infty(\mathbb{N}, \mathbb{C})$ by choosing $\widehat{g}(1) \in \mathbb{C}$ such that $|\widehat{g}(1)| \le \|g\|_\infty = \|\varphi\|$. Now $\widehat{\varphi}(f) = \sum_{k=1}^\infty f(k)\widehat{g}(k)$ extends φ to V and satisfies $\|\widehat{\varphi}\| = \|\widehat{g}\|_\infty = \|g\|_\infty = \|\varphi\|$. In view of $\|\varphi\| > 0$, we clearly have no uniqueness of the norm-preserving extension $\widehat{\varphi}$.

Solution 2 (i) Let $S \subseteq V^*$ be a countable dense subset. We may assume $0 \notin S$. For each $\varphi \in S$ pick $x_\varphi \in V$ such that $\|x_\varphi\| = 1$ and $|\varphi(x_\varphi)| \ge \|\varphi\|/2$. Put $T = \{x_\varphi \mid \varphi \in S\}$ and $Y = \text{span}_{\mathbb{Q}}(T)$ or $Y = \text{span}_{\mathbb{Q}+i\mathbb{Q}}(T)$, depending on $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Assume $\psi \in V^*$ satisfies $\psi(x) = 0\ \forall x \in Y$. Let $\varepsilon > 0$. Since $S \subseteq V^*$ is dense, there exists $\varphi \in S$ such that $\|\psi - \varphi\| < \varepsilon$. Thus $|\varphi(x_\varphi)| = |\psi(x_\varphi) - \varphi(x_\varphi)| < \varepsilon$ since $x_\varphi \in Y \subseteq \ker\psi$ and $\|x_\varphi\| = 1$. In view of the choice of x_φ, this implies $\|\varphi\|/2 \le |\varphi(x_\varphi)| < \varepsilon$. Thus $\|\varphi\| < 2\varepsilon$ and therefore $\|\psi\| \le 3\varepsilon$. Since ε was arbitrary, this implies $\psi = 0$. Thus only the zero functional vanishes on Y, in other words $Y^\perp = \{0\}$. By Exercise 8.9, this is equivalent to $\overline{Y} = V$. Since Y is countable, V is separable.

(ii) If V is reflexive then $V^{**} \cong V$ is separable. Now separability of V^* follows from (i).

(iii) If S is infinite then the power set $P(S)$ is uncountable. If $T, T' \subseteq S$ and $T \ne T'$ then $\|\chi_T - \chi_{T'}\|_\infty = 1$. If $U \subseteq L^\infty(S, \mathbb{F})$ is dense then we have $\bigcup_{f\in U} B(f, \varepsilon) = L^\infty(S, \mathbb{F})$ for each $\varepsilon > 0$. If $\varepsilon < 1/2$, each ball contains only one of the χ_T, so that U must be uncountable. (A diagonal argument is also OK.)

(iv) If $1 < p < \infty$ and $S = \mathbb{N}$ then $V = \ell^p(S, \mathbb{F})$ is infinite dimensional and separable with separable V^* (by Theorem 5.16(ii) and Proposition 5.14). On the other hand, $V = \ell^1(S, \mathbb{F})$ is separable, while $V^* \cong \ell^\infty(S, \mathbb{F})$ is not by (iii).

Solution 3 (i) Let $Q = V/W$ with quotient map $p : V \to Q$. Since Z is finite dimensional, so is $p(Z) \subseteq Q$. Thus $p(Z)$ is closed by Exercise 4.22, thus also $p(W + Z) = p(Z)$. Now continuity of p implies that $W + Z = p^{-1}(p(W + Z)) \subseteq V$ is closed.

(ii) By standard linear algebra, there exists a linear subspace $Z \subseteq W$ that is algebraically complementary, i.e. $Z \cap AV = \{0\}$ and $Z + AV = W$, so that $W = AV \oplus Z$ algebraically. It satisfies $\dim Z = \dim(W/AV) < \infty$ and therefore is automatically closed, thus Banach. Now $V \oplus Z$, equipped with the norm $\|(x, z)\| = \|x\| + \|z\|$, is a Banach space. Define a linear map $\alpha : V \oplus Z \to W$, $(x, z) \mapsto Ax + z$. In view of

$$\|z + Ax\| \le \|z\| + \|A\|\|x\| \le (1 + \|A\|)\|(x, z)\|,$$

α is bounded. And it is a bijection by construction (since $W = AV \oplus Z$). Thus by the Bounded Inverse Theorem α is a homeomorphism. Since $V \oplus 0 \subseteq V \oplus Z$ is a closed subspace, $AV = \alpha(V \oplus 0) \subseteq W$ is closed, proving the claim.

(iii) ker$A$ is closed by continuity of A. Thus by Proposition 7.2, $A/\ker A$ is a Banach space and there is an $A' \in B(V/\ker A, W)$ such that $A = \widetilde{A} \circ p$, where $p : V \to V/\ker A$ is the quotient map. Since A' is injective and $AV = A'(V/\ker A)$, the claim follows from (ii).

Solution 4 We have $(c^*c)^* = c^*c^{**} = c^*c$, thus c^*c is self-adjoint. By Remark 12.22.1 we have $c = a + ib$ with a, b self-adjoint, and normality of c is equivalent to $ab = ba$. Using this, we have $c^*c = (a+ib)^*(a+ib) = (a-ib)(a+ib) = a^2 + b^2$. Since a^2, b^2 are positive by Proposition 13.10(i) and commute with each other, $c^*c = a^2 + b^2$ is positive by Exercise 11.49(i).

Solution 5 In this proof I write A instead of A_λ.

(i) Assume A is bounded. Then $|\lambda_n| = \|\lambda_n \delta_{n+1}\| = \|A\delta_n\| \leq \|A\|\|\delta_n\| = \|A\| \ \forall n$, thus $\{\lambda_n\}$ is bounded. Assume $\{\lambda_n\}$ is bounded, i.e. $|\lambda_n| \leq M \ \forall n$, and let $f \in \ell^2(\mathbb{N}, \mathbb{C})$. Then $f = \sum_{n=1}^\infty f(n)\delta_n$, so that $Af = \sum_{n=1}^\infty f(n)\lambda_n\delta_{n+1}$. Since the δ_n are orthonormal, this gives $\|Af\|^2 = \sum_{n=1}^\infty |\lambda_n f(n)|^2 \leq M^2 \sum_{n=1}^\infty |f(n)|^2 = M^2\|f\|^2$, so that $\|A\| \leq M$.

If $\lambda_n = 0$ then $A\delta_n = \lambda_n\delta_{n+1} = 0$, while $\delta_n \neq 0$, so that A is not injective. For the converse, assume $\lambda_n \neq 0 \ \forall n$ and $Af = Ag$. Then $Af - Ag = \sum_{n=1}^\infty \lambda_n(f(n) - g(n))\delta_{n+1} = 0$. Since the δ_n are orthonormal and $\lambda_n \neq 0$, this implies $f = g$.

We have $A^n\delta_k = \lambda_k\lambda_{k+1}\cdots\lambda_{k+n-1}\delta_{k+n}$. By definition, A is nilpotent if and only if $A^n = 0$ for some $n \in \mathbb{N}$. This is equivalent to $\prod_{i=0}^{n-1} \lambda_{k+i} = 0 \ \forall k$. Thus A is nilpotent precisely if there is a finite upper bound on the length of contiguous non-zero subsequences of $\{\lambda_k\}$.

(ii) All λ_k are non-zero, thus A is injective by (i). Thus all powers of A are injective, so that A is not nilpotent. (This also follows from (i).)

We have $A^n\delta_k = 2^{-k-(k+1)-\cdots-(k+n-1)}\delta_{k+n} = 2^{-nk-(n-1)n/2}\delta_{k+n}$. If $x \in H$ then

$$\|A^n x\|^2 = \sum_{k=1}^\infty |2^{-nk-(n-1)n/2}\, x(k)|^2 \leq 2^{-2n-(n-1)n} \sum_{k=1}^\infty |x(k)|^2.$$

Thus $\|A^n\| \leq 2^{-n-(n-1)n/2} = 2^{-n(n+1)/2}$, so that $\|A^n\|^{1/n} \leq 2^{-(n+1)/2}$, which converges to zero as $n \to \infty$. By the spectral radius formula we have $r(A) = 0$, so that A is quasi-nilpotent.

(iii) We have

$$\langle A\delta_k, \delta_\ell\rangle = \langle \lambda_k\delta_{k+1}, \delta_\ell\rangle = \lambda_k\delta_{k+1,\ell} = \langle \delta_k, A^*\delta_\ell\rangle,$$

which readily gives $A^*\delta_1 = 0$ and $A^*\delta_\ell = \overline{\lambda_{\ell-1}}\delta_{\ell-1}$ if $\ell \geq 2$. Thus $A^*A\delta_k = A^*\lambda_k\delta_{k+1} = |\lambda_k|^2\delta_k \ \forall k$. Now $|A| = \sqrt{A^*A}$ is the diagonal operator that multiplies each δ_k by $|\lambda_k|$. Since the polar decomposition is $A = V|A|$, it is clear that V must multiply each δ_k by $\text{sgn}(\lambda_k)$ [recall that $\text{sgn}(0) = 0$ and $\text{sgn}(z) = z/|z|$ if $z \neq 0$] and shift to the right: $V\delta_k = \text{sgn}(\lambda_k)\delta_{k+1}$. (Thus $V = A_{\text{sgn}\,\lambda}$.) It is evident that V is an isometry. That this V is the one given by polar decomposition follows from $\ker V = \overline{\text{span}_\mathbb{C}\{\delta_k \mid \lambda_k = 0\}} = \ker A$ and the uniqueness part of the polar decomposition.

(iv) As we know from Exercise 15.11, $|A|$ is compact if and only if $\{\lambda_k\} \in c_0(\mathbb{N}, \mathbb{F})$, to wit $\lambda_k \to 0$. That compactness of A implies compactness of $|A|$ is proven in the notes: It follows from $|A| = V^*A$, where $A = V|A|$ is the polar decomposition. It is equally obvious that compactness of $|A|$ implies compactness of A. Thus A is compact if and only if $\lambda_k \to 0$.

(v) As shown above, $A^*A\delta_k = |\lambda_k|^2\delta_k \ \forall k$. Similarly, $AA^*\delta_1 = 0$, while for $k \geq 2$ we have $AA^*\delta_k = A\overline{\lambda}_{k-1}\delta_{k-1} = |\lambda_{k-1}|^2\delta_k$. It follows that A is normal if and only if $\lambda_1 = 0$ and $|\lambda_k| = |\lambda_{k-1}|$ for all $k \geq 2$. Now a trivial induction shows that A is normal if and only if $\lambda_k = 0 \ \forall k$, to wit $A = 0$. Thus the spectral theorems of Chapter 16 and Theorem 15.27 apply only in the trivial case $A = 0$. But Proposition 15.29 does apply whenever $\lambda_k \to 0$. Since $A = \sum_{k=1} \lambda_k\delta_{k+1}\langle\cdot, \delta_k\rangle$ by its very definition, it is a triviality to find orthonormal sets E, F and numbers β_e as in that proposition.