EXAM MANIFOLDS
APRIL 1 2019, 8:30-11:30

-You are not allowed to use books, notes, notebooks, mobile phones, tablets, etc.
-If you are leaving temporarily the room, please hand in your mobile phone to the supervisors.
-For each question, you are expected to provide complete arguments; YES/NO do not suffice.
-Please write the letter and the number of the (sub)problem you are solving (e.g."Problem C. (2)")
-Write the answers either in English or in Dutch.
-Don't forget to ***write your name*** *on each sheet of paper you are handing in.*

Problem A. *(2 pts)* Consider the stereographic projection on the 2-sphere

$$\pi_n : S^2\backslash\{n\} \longrightarrow \mathbb{C}, \qquad \pi_n(a,b,c) = \frac{a+\mathbf{i}b}{1-c} \in \mathbb{C},$$

where $n = (0,0,1)$ and we identify $\mathbb{C} \cong \mathbb{R}^2$, via the map $z = x + iy \mapsto (x, y)$.

Show that for any complex polynomial $f \in \mathbb{C}[z]$ there exists a unique smooth function $\underline{f} : S^2 \to S^2$ which satisfies on $S^2\backslash\{n\}$ the equality:

$$f \circ \pi_n = \pi_n \circ \underline{f}.$$

Problem B. *(2 pts)* On $\mathbb{R}^2$ consider the vector fields

$$X = \frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \quad Y = \frac{\partial}{\partial x}$$

and the differential one-form

$$\omega = xdy - ydx.$$

(1) Calculate $[X, Y]$ and $d\omega$ and $\iota_X\omega$.
(2) Calculate the flow X on $\mathbb{R}^2$; i.e. $\phi^t_X(x, y)$, for $(x, y) \in \mathbb{R}^2$.
(3) Check the following formula through a direct calculation:

$$\frac{d}{dt}(\phi^t_X)^*(\omega)\big|_{t=0} = d\iota_X\omega + \iota_X d\omega.$$

Problem C. *(2 pts)* Let $f : M \to N$ be a surjective submersion between smooth manifolds.
(1) Show that f is an open map (i.e. $f(U)$ is open, for every open set $U \subset M$).
(2) Show that for any vector field $Y \in \mathfrak{X}(N)$ there exists a vector field $X \in \mathfrak{X}(M)$ which is f-related to Y, i.e.

$$d_pf(X_p) = Y_{f(p)}, \quad \text{for all } p \in M.$$

Hint: use a partition of unity.

Problem D. *(2 pts)* We regard the 2-torus $T := S^1 \times S^1 \subset \mathbb{C}^2$, as a Lie group with multiplication $(z_1, z_2)\cdot(w_1, w_2) = (z_1 \cdot w_1, z_2 \cdot w_2)$. Denote the left translation by $g \in T$ by

$$\lambda_g : T \to T, \quad \lambda_g(h) = gh.$$

(1) A one-form $\alpha \in \Omega^1(T)$ is called **left invariant** if it satisfies

$$\lambda_g^*(\alpha) = \alpha, \quad \text{for all } \ g \in T.$$

Show that every left invariant one-form on T is closed.
Hint: write this condition in "angle coordinates" $\theta_1, \theta_2 \in \mathbb{R}/2\pi\mathbb{Z}$ on T.

(2) Let α be a closed 1-form on T. Show that the integral

$$\int_{S^1\times\{z\}} \alpha$$

is independent of $z \in S^1$ (where we use the counterclockwise orientation on $S^1 \times \{z\}$).

Problem E. *(2 pts)* Let M be a 2-dimensional smooth manifold. Let $p \in M$, and let $X \in \mathfrak{X}(M)$ be a vector field such that $X_p \neq 0$.
(1) Show that there exists a smooth curve $\gamma : (-\epsilon, \epsilon) \to M$, with $\gamma(0) = p$, such that the vectors $v := \gamma'(0)$ and $w := X_p$ form a basis of T_pM.
(2) Let ϕ^t_X denote the flow of X and let γ be as in item (1). Show that the map $\varphi(t, s) = \phi^t_X(\gamma(s))$ defines a diffeomorphism between a neighborhood of $(0, 0)$ in $\mathbb{R}^2$ and a neighborhood of p in M.
(3) Show that there exists a smooth chart $(U, (x^1, x^2))$ around p in which $X|_U = \frac{\partial}{\partial x^1}$.