EXAM MANIFOLDS
JANUARY 18 2019, 8:30-11:30

Some remarks:
-You are not allowed to use books, notes, notebooks, mobile phones, tablets, etc.
-When you are leaving temporarily the room during the exam, please hand in your mobile phone to the supervisors.
-For each question, you are expected to provide complete arguments; just YES/NO answers are worth zero points.
-Please write both the letter of the problem and the number of the subproblem you are solving (for example:"Problem C. (2)"*).*
-Write the answers either in English or in Dutch.
-Don't forget to ***write your name*** *on each sheet of paper you are handing in.*
-Good luck!!

Problem A. *(2 pts)* Consider the vector fields on $\mathbb{R}^3$:

$$X = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \quad Y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}.$$

(1) Calculate the Lie bracket $[X, Y]$.
(2) Calculate the flow of X on $\mathbb{R}^3$.

Problem B. *(2 pts)* On the 2-torus $T^2 = S^1 \times S^1$ consider the angle coordinates $\theta_1, \theta_2 \in \mathbb{R}/2\pi\mathbb{Z}$.
(1) Give a smooth atlas on T^2.
(2) Consider the map $f : T^2 \longrightarrow \mathbb{R}^3$,

$$f(\theta_1, \theta_2) = (2 + \cos\theta_2)(\cos\theta_1, \sin\theta_1, 0) + (0, 0, \sin\theta_2).$$

Show that f is an immersion. Is f an embedding?

Problem C. *(2 pts)*
(1) Let V and W be finite dimensional real vector spaces, and let $A : V \to W$ be a surjective linear map. Let o_V be an orientation on V and o_W be an orientation on W. Prove that the following formula defines an orientation o_K on $K := \ker(A)$:

$$o_K(v_1, \ldots, v_k) := o_V(v_1, \ldots, v_k, v_{k+1}, \ldots, v_m) o_W(A(v_{k+1}), \ldots, A(v_m)),$$

where $(v_1, \ldots, v_m)$ is a basis of V so that the first k vectors $(v_1, \ldots, v_k)$ form a basis of K.
(2) Let $f : M \to N$ be a smooth map between orientable manifolds M and N. If $p \in N$ is a regular value of f, prove that $f^{-1}(p)$ is an orientable manifold.

Problem D. *(2 pts)* Let $\alpha \in \Omega^1(M)$ be a 1-form which is nowhere zero:

$$\alpha_x \neq 0, \quad \text{for all} \quad x \in M.$$

Prove that
(1) There exists a vector filed X on M such that $\iota_X \alpha = 1$ (*Hint: Use a partition of unity*).
(2) For any differential form $\beta \in \Omega^k(M)$ the following are equivalent (*Hint: Use (1)*):
  (a) $\alpha \wedge \beta = 0$;
  (b) there exists $\gamma \in \Omega^{k-1}(M)$ such that $\beta = \alpha \wedge \gamma$.

Problem E. *(2 pts)* Let $M := \mathbb{R}^2 - \{(0, 0)\}$. Consider the 1-form on M given by

$$\omega := \frac{xdy - ydx}{x^2 + y^2} \in \Omega^1(M).$$

For a smooth map $\gamma : S^1 \to M$ define

$$\mathcal{W}(\gamma) := \int_{(S^1, o)} \gamma^*(\omega),$$

where S^1 is endowed with the orientation o such that $o(\frac{\partial}{\partial\theta}) = 1$.
(1) Show that $d\omega = 0$.
(2) Let $\gamma : S^1 \times [0, 1] \to M$ be a smooth function, which we think of as a smooth family $\gamma_t : S^1 \to M$, for $t \in [0, 1]$. Show that

$$\mathcal{W}(\gamma_0) = \mathcal{W}(\gamma_1).$$