Advanced Probability 2018
Final Examination

29 October 2018, 12:30 – 14:30

- The total number of points available is 100. Full points will only be awarded with full justification for your solutions.
- Write solutions on separate sheets, ≤ 1 problem per sheet, with name and student number written at the top of each sheet. Sheets without proper identification may be discarded.
- Memory aids: $\frac{\lambda^k e^{-\lambda}}{k!}$ (Poisson), $\lambda e^{-\lambda x}$ (exponential), $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ (standard normal), $\mathbb{E}(e^{tX})$ (m.g.f.).

Problem 1 (30 points).

Lucy is at a lemonade stand with three cups of lemonade. Customers arrive according to a Poisson distribution of rate 1/4 (per hour). Each customer buys and drinks exactly one cup of lemonade, then leaves. Let T be how long (in hours) until Lucy sells all her lemonade.

(a) Prove that T is distributed as the sum of three independent exponential r.v.'s of rate 1/4.

(b) Prove that T has density function $f(x) = \frac{x^2}{2^7} e^{-x/4}, x > 0$.

See next page.

Problem 2 (40 points).

(a) Show $\Pr(X \geq a) \leq \mathbb{E}(X)/a$ for a non-negative r.v. X and $a > 0$. (Hint: use $\int_a^\infty x f(x)\,dx \geq \int_a^\infty a f(x)\,dx$, where f is the p.d.f.)

(b) Show that the moment generating function of a Bernoulli r.v. of parameter p is $1 - p + pe^t$.

(c) Let Y be a binomial r.v. of parameters n and p. Show that, if $c > p$, then $\Pr(Y \geq cn) \leq \exp(-nI(c))$, where $I(c) = \sup\{ct - \ln(1 - p + pe^t) \,:\, t > 0\}$.
(Hint: explain why $A \geq B$ iff $e^{tA - tB} \geq 1$, then use (a) and (b).)

Problem 3 (30 points).

The Cauchy distribution has density function $f(x) = \frac{1}{\pi(1+x^2)}$. Let Y be the ratio of two independent standard normal r.v.'s.

(a) Show that Y has density function $\int_{-\infty}^{\infty} |y|\, g(uy, y)\, dy$ for some appropriately chosen function g.

(b) Evaluate the integral to show that Y has a Cauchy distribution.

The end!