Problem A: $f \in \mathbb{C}[z] \Rightarrow \exists! \underline{f} \in C^\infty(S^2): \quad f \circ \pi_n = \pi_n \circ \underline{f} \quad$ on $S^2 \setminus \{N\}$ $(*)$

i) $(*)$ implies smoothness of $\underline{f}$ on $S^2 \setminus \{N\}$ since the representative of $\underline{f}$ in the chart π_n is given by f

ii) Check $\underline{f}$ extends smoothly over N: representative of $\underline{f}$ in π_s is: $\pi_s \circ \underline{f} \circ \pi_s^{-1}$

on $\mathbb{C} \setminus \{0\}$ this is already determined by $(*)$:

$$\pi_s \circ \underline{f} \circ \pi_s^{-1}\Big|_{\mathbb{C}\setminus\{0\}} = \tau \circ \pi_n \circ \underline{f} \circ \pi_s^{-1}\Big|_{\mathbb{C}\setminus\{0\}} = \tau \circ f \circ \pi_n \circ \pi_s^{-1}\Big|_{\mathbb{C}\setminus\{0\}} = \tau \circ f \circ \tau^{-1}\Big|_{\mathbb{C}\setminus\{0\}} \quad \text{w/ } \tau = \pi_s \circ \pi_n^{-1}$$

and $\tau^{-1} = \tau$

hence we have to check if $\tau \circ f \circ \tau$ extends smoothly over 0: $\quad (\tau(z) = \frac{1}{\bar{z}})$

$$\tau \circ f \circ \tau(z) = \frac{1}{\bar{f}(\frac{1}{z})}$$

Note: $\tau \circ p(z) = \frac{1}{\bar{p}(\bar{z})}$ for $p \in \mathbb{C}[z]$

Case 1: $\deg f = 0 \rightsquigarrow f$ is a constant $\rightsquigarrow \tau \circ f \circ \tau$ extends smoothly

Case 2: $\deg f = d$ then: $\frac{1}{\bar{f}(z)} = \frac{1}{\sum_{i=0}^{d} \bar{a}_i \frac{1}{z^i}} = \frac{1}{\frac{1}{z^d}\sum_{i=1}^{d} \bar{a}_i z^{d-i}} = \frac{z^d}{\bar{a}_d + \sum_{i=0}^{d-1} \bar{a}_i z^{d-i}}$

which is a holomorphic function around 0 hence smooth

Uniqueness of $\underline{f}$ follows because $(*)$ defines $\underline{f}$ on an open dense set and continuity of $\underline{f}$

Problem B: $X = \partial_x - x\partial_y$, $Y = \partial_x$, $\omega = x dy - y dx$

1) $\cdot)$ $[X, Y] = X(Y) - Y(X) = 0 - \partial_x(-x\partial_y) = \partial_y$

$\cdot)$ $d\omega = d(x dy - y dx) = 2 dx \wedge dy$

$\cdot)$ $i_X \omega = \omega(X) = x\, dy(X) - y\, dx(X) = -x^2 - y$

2) The flow ϕ_X^t satisfies: $\frac{d}{dt}\phi_X^t(x,y) = X(\phi_X^t(x,y)) \iff \begin{array}{l}\dot{x}(t) = 1 \\ \dot{y}(t) = -x(t)\end{array}$

w/ solutions: $x(t) = t + c_1$, $y(t) = -\int_0^t x(s)ds + c_2 = -\frac{t^2}{2} - tc_1 + c_2$

$\phi_X^0(x_0, y_0) = (x_0, y_0) \iff (x(0), y(0)) = (x_0, y_0)$

hence: $c_1 = x_0$, $c_2 = y_0$

and the flow $\phi_X^t(x_0, y_0) = (t + x_0, -\frac{t^2}{2} - tx_0 + y_0)$

Problem B

3) $\cdot$) $d\, \iota_X \omega = d(-x^2 - y) = -2x\,dx - dy$

$\cdot$) $\iota_X d\omega = \iota_X(2\,dx \wedge dy) = 2\,dx(X)\,dy - 2\,dy(X)\,dx = 2\,dy + 2x\,dx$

hence $d\,\iota_X\omega + \iota_X d\omega = dy$

$\cdot$) $(\phi_t^* \omega)(x,y)(t) = (t+x)(-t\,dx + dy) + \left(\frac{t^2}{2} + tx + y\right) dx$

hence $\frac{d}{dt}(\phi_t^*\omega)\Big|_{t=0}(x,y) = -t\,dx + dy - (t+x)\,dx + (t+x)\,dx\Big|_{t=0} = dy$

Problem C

1) By the local submersion thm we get that around each point $p \in M$

f is an open map for small enough open nbhd's of p by the topology of $\mathbb{R}^m, \mathbb{R}^n$ respect.

Let $U \subset M$ be an arbitrary open set, pick for all $p \in U$ such a open nbhd U_p

$\rightsquigarrow$ $U = \bigcup_{p \in U} (U_p \cap U)$ and $f(U) = \bigcup_{p \in U} f(U_p \cap U)$ is a union of open sets hence open

2) Let $Y \in \mathfrak{X}(N)$ by a vector field, pick an atlas A_M and an atlas A_N such that for each $(U_i, \varphi_i) \in A_M$ $\exists (V_j, \psi_j) \in A_N$ satisfying the local subm. thm.

Choose a partion of unity according to A_M $\{\chi_i\}$

on (U_i, φ_i) it is easy defining such a X_i using the local subm. thm.

then define $X := \chi_i X_i$

Problem D

1) Let $g = (e^{i\varphi_1}, e^{i\varphi_2})$, (θ_1, θ_2) local coordinates on T

now $\lambda_g(\theta_1, \theta_2) = (\theta_1 + \varphi_1, \theta_2 + \varphi_2)$

hence $\lambda_g^*(\alpha) = \alpha$ means locally for $\alpha = a_1(\theta_1, \theta_2)\,d\theta_1 + a_2(\theta_1, \theta_2)\,d\theta_2$

$$a_1(\theta_1 + \varphi_1, \theta_2 + \varphi_2)\,d\theta_1 + a_2(\theta_1 + \varphi_1, \theta_2 + \varphi_2)\,d\theta_2 = a_1(\theta_1, \theta_2)\,d\theta_1 + a_2(\theta_1, \theta_2)\,d\theta_2$$

$\Rightarrow$ $a_j(\theta_1 + \varphi_1, \theta_2 + \varphi_2) = a_j(\theta_1, \theta_2)$ so the a_j's are actually constant

and therefore α is closed

Problem D

2) Let $\alpha \in \Omega^1(T)$ be closed and $z, w \in S^1$

then: $\int_{S^1\times\{z\}} \alpha - \int_{S^1\times\{w\}} \alpha = \int_{(\partial M, \partial_0)} \alpha \overset{\text{Stoke's thm.}}{=} \int_{M,\theta} d\alpha = 0$

where $M = S^1 \times \{e^{i(\Theta+\Theta_0)}\}$, $e^{i\Theta_0} = w$, $e^{i\Theta_1} = z$, $0 \leq \Theta \leq \Theta_1 - \Theta_0$

and θ the orientation induced from the orientation on T

Problem E

1) Let $p \in M$, $X \in \mathfrak{X}(M)$ and pick a chart (u, φ) around p

$d\varphi_p(X_p) =: \tilde{w}$ pick a linear indep vector $\hat{v} \in T_p\varphi(u) \cong \mathbb{R}^2$

and define $\hat{\gamma}(t) := \varphi(p) + t\hat{v}$ and $\gamma(t) := \varphi^{-1}(\hat{\gamma}(t))$

which is inside $\varphi(u)$ for small t

then $\dot{\gamma}(0)$ and X_p are linear indep. since φ^{-1} is a local diffeom.

2) First note $\Psi(t,s) = \phi_X^t(\gamma(s))$ is smooth for small t,s as it is a composition of smooth maps

Moreover $\Psi(0,0) = \phi_X^0(\gamma(0)) = \phi_X^0(p) = p$

and finally: i) $\partial_t \Psi(t,s)\big|_{(0,0)} = \partial_t \Psi(t,0)\big|_{t=0} = X(\phi_X^t(p))\big|_{t=0} = X_p = w$

ii) $\partial_s \Psi(t,s)\big|_{(0,0)} = \partial_s \Psi(0,s)\big|_{s=0} = \partial_s \gamma(s)\big|_{s=0} = v$

hence the differential of Ψ is an isomorphism at $(0,0)$ and therefore Ψ defines a local diffeom around $(0,0)$ by the local diffeom thm

3) By i) in 2) we see: $d\Psi(\partial_t) = X$

hence let $0 \in V \subset \mathbb{R}^2$ be such that $\Psi|_V$ is a diffeom. and define the chart (u, Ψ)

by: $u := \Psi(V)$, $\Psi := (\Psi|_V)^{-1}$