Solutions Manifolds exam
FALL 2018

Problem A (1) $[X, Y] = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$.

(2) $\phi_X^t : \mathbb{R}^3 \to \mathbb{R}^3$

$$\phi_X^t(x_0, y_0, z_0) = (x_0, \cos(t)y_0 - \sin(t)z_0, \sin(t)y_0 + \cos(t)z_0)$$

Problem B (1) An atlas on $S^1 \times S^1$ can be obtained as follows:

$$\mathcal{A}_{S^1\times S^1} = \{ (U\times V, \varphi\times\psi), (U,\varphi),(V,\psi)\in\mathcal{A}_{S^1}\},$$

where $\mathcal{A}_{S^1}$ is a fixed atlas on S^1.

For example, one can take $\mathcal{A}_{S^1}$ to be the stereographic atlas on S^1.

(2) Let $v = a\frac{\partial}{\partial\theta_1} + b\frac{\partial}{\partial\theta_2} \in T_{(\theta_1,\theta_2)}(S^1\times S^1)$.

$$w := (d_{(\theta_1,\theta_2)} f)(v) = a\cdot(2+\cos\theta_2)(-\sin\theta_1, \cos\theta_1, 0) + b(-\sin\theta_2)(\cos\theta_1, \sin\theta_1, 0) + b\cdot(0, 0, \cos\theta_2).$$

Note that $|w|^2 = a^2(2+\cos\theta_2)^2 + b^2$.

Thus $w = 0 \iff a = b = 0$.

~~So $d_{(\theta_1,\theta_2)} f$ is injective~~

So $d_{(\theta_1,\theta_2)} f$ is injective $\Rightarrow$ f is an immersion.

We show that f is injective:

$$f(\theta_1,\theta_2) = f(\varphi_1,\varphi_2) \overset{3^{rd} \text{ coordinate}}{\Longrightarrow} \sin\theta_2 = \sin\varphi_2$$

$$|f|^2 = \dots \Longrightarrow \dots \Rightarrow \cos\theta_2 = \cos\varphi_2$$

$$\left.\begin{matrix} \sin\theta_2 = \sin\varphi_2 \\ \cos\theta_2 = \cos\varphi_2 \end{matrix}\right\} \theta_2 = \varphi_2$$

$$\Rightarrow (\cos\theta_1, \sin\theta_1) = (\cos\varphi_1, \sin\varphi_1) \Rightarrow \theta_1 = \varphi_1$$

Thus, f is injective.

Any injective immersion from a compact manifold to any other manifold is an embedding.

Thus f is an embedding

Problem C (1).

We need to show that

Claim: O_k is well-defined, i.e. if $w_{k+1}, \dots, w_m \in V$ are s.t. $v_1, \dots, v_k, w_{k+1}, \dots, w_m$ are a basis of V, then

$$O_V(v_1, \dots, v_m) \cdot O_W(A(v_{k+1}), \dots, A(v_m)) =$$

$$= O_V(v_1, \dots, v_k, w_{k+1}, \dots, w_m) \cdot O_W(A(w_{k+1}), \dots, A(w_m))$$

Proof: Let $B : V \xrightarrow{\sim} B$ be defined s.t.

$$\begin{cases} B v_1 = v_1 \\ \vdots \\ B v_k = v_k \\ B v_{k+1} = w_{k+1} \\ \\ B v_m = w_m \end{cases}$$

In the basis $v_1, \dots, v_m$, B has matrix

$$[B] = \begin{pmatrix} I & [\cancel{A}] \\ 0 & [C] \end{pmatrix}$$

$$
\begin{aligned}
o_V(v_1,\dots,v_k,w_{k+1},\dots,w_m) &= o_V(Bv_1,\dots,Bv_k,Bv_{k+1},\dots,Bv_m) = \\
&= \mathrm{sign}(\det B)\cdot o_V(v_1,\dots,v_m) = \\
&= \mathrm{sign}(\det[C])\cdot o_V(v_1,\dots,v_m)
\end{aligned}
\qquad \mathrm{I}.
$$

On W we have the bases:

$$
\left.\begin{array}{l} A v_{k+1},\dots, A v_m \\ \text{and} \\ A w_{k+1},\dots, A w_m \end{array}\right\} \ (*)
$$

$k+1 \le i \le m$: $W_i = B v_i = \sum_{j=k+1}^{m} c_{ij} v_j + \sum_{j=1}^{k} d_{ij} v_j$

Since $A v_j = 0$ $1 \le j \le k$;

$\Rightarrow A w_i = A B v_i = \sum_{j=k+1}^{m} c_{ij} A v_j$ &

$\Rightarrow$ the change of basis map between the bases in $(*)$ has determinant $= \det[C]$.

$\Rightarrow$ $o_W(Aw_{k+1},\dots,Aw_m) = \mathrm{sign}(\det C)\cdot o_W(Av_{k+1},\dots,Av_m)$ $\quad \mathrm{II}.$

$\mathrm{I}. + \mathrm{II}.$ imply Claim 1.

Claim 2: o_K is an orientation.

This follows directly because o_V is an orientation.

Problem C (2)

By the regular value theorem, $f^{-1}(p)$ is an embedded submanifold of M with:

$$T_q(f^{-1}(p)) = \mathrm{Ker}(d_q f).$$

Let σ_M be an orientation on M
and σ_N be an orientation on N.
Define the following orientation on $P := f^{-1}(p)$:

$$\sigma_P(v_1,\dots,v_k) := \sigma_M(v_1,\dots,v_k,v_{k+1},\dots,v_m)\,\sigma_N(d_pf(v_{k+1}),\dots,d_pf(v_m))$$

where $v_1,\dots,v_k \in T_q P = \ker d_q f$ is a basis

and $v_1,\dots,v_k,v_{k+1},\dots,v_m$ is a basis of $T_q M$.

By (1) $\sigma_P(q)$ is an orientation of $T_q P$ $\forall q \in P$.

To see that σ_P is a smooth orientation, we apply the submersion theorem: ~~In a pair chart:~~ In a pair of charts, we have

$$\cancel{f(x_1,\dots,x_m) = (x_1,\dots,x_n)}$$

$$f(x_1,\dots,x_m) = (x_{k+1},\dots,x_m)$$

We may assume that

$$\sigma_M\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_m}\right) = 1$$

$$\sigma_N\left(\frac{\partial}{\partial x_{k+1}},\dots,\frac{\partial}{\partial x_m}\right) = 1$$

$$\Rightarrow \sigma_P\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_k}\right) = 1$$

This shows that σ_P is smooth.

Problem D (1)

Let $(U, x^1, \ldots, x^m)$ be a chart on M

Write $\alpha|_U = \sum_{i=1}^{m} a_i \, dx^i$

Define $X^U := \frac{1}{a_1^2 + \cdots + a_m^2} \left(\sum_{i=1}^{m} a_i \frac{\partial}{\partial x^i} \right)$

~~Then~~ Since ~~$\alpha|_U$~~ $\alpha_p \neq 0 \ \forall p \in U$

$\Rightarrow \quad a_1^2 + \cdots + a_m^2 > 0$.

Hence X^U is a smooth vector field on U.

Clearly $i_{X^U} \alpha|_U = 1$.

Let $\{(U^i, \varphi^i)\}_{i \in I}$ be an atlas on M.

For each $i \in I$ ~~let X^i s~~

let X^i be a vector field on U^i s.t.

$i_{X^i} \alpha|_{U^i} = 1$

Let $\{\rho^i\}_{i \in I}$ be a partition of unity

subordinated to $\{U^i\}_{i \in I}$.

Define $X := \sum_{i \in I} \rho^i X^i$.

Then $i_X \alpha = \sum_{i \in I} \underbrace{\rho^i (i_{X^i} \alpha)}_{= \rho^i} = \sum_{i \in I} \rho^i = 1$

Problem D (2)

a ⇒ b:

$$\alpha \wedge \beta = 0$$

Let X be as in (1)

$$\Rightarrow \quad i_X \underbrace{(\alpha \wedge \beta)}_{=0} = \underbrace{(i_X \alpha)}_{=1} \wedge \beta - \alpha \wedge (i_X \beta)$$

$$\Rightarrow \quad \boxed{\alpha \wedge i_X \beta = \beta}$$

So for $\gamma := i_X \beta$ we have that

$$\beta = \alpha \wedge \gamma .$$

b ⇒ a: $\alpha \wedge (\alpha \wedge \gamma) = \underbrace{(\alpha \wedge \alpha)}_{=0} \wedge \gamma = 0$

Problem E (1)

$$d\omega = \left(\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right) dx \wedge dy =$$

$$= \left(\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}\right) dx \wedge dy =$$

$$= \left(\frac{2}{x^2+y^2} - \frac{2(x^2+y^2)}{(x^2+y^2)^2}\right) dx \wedge dy = 0.$$

(2). Solution 1: By the homotopy invariance of de Rham cohomology, we know that:

$$\gamma_0^*[\omega] = \gamma_1^*[\omega] \quad \text{in } H^1(S^1)$$

$$\Leftrightarrow \quad \gamma_0^*\omega = \gamma_1^*\omega + df, \text{ for some } f \in C^\infty(S^1)$$

$$\Rightarrow \int_{(S^1,o)} \gamma_0^*\omega = \int_{(S^1,o)} \gamma_1^*\omega + \int_{(S^1,o)} df$$

$$\int_{(S^1,o)} df \underset{\uparrow}{=} \int_{\partial S^1} f = 0$$

Stokes = Fund. Theorem of Calculus.

$$\Rightarrow W(\gamma_0) = W(\gamma_1).$$

Solution 2: We apply Stokes to the manifold with boundary $S^1 \times [0,1]$. For the standard orientation

$$o_{S^1\times[0,1]}\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial t}\right) = 1,$$

the induced ~~the~~ boundary orientation is:

$$\sigma_{S^1\times\{0\}}\left(\frac{\partial}{\partial\theta}\right) = \sigma_{S^1\times[0,1]}\left(-\frac{\partial}{\partial t}, \frac{\partial}{\partial\theta}\right) = 1$$

$$\sigma_{S^1\times\{1\}}\left(\frac{\partial}{\partial\theta}\right) = \sigma_{S^1\times[0,1]}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial\theta}\right) = -1$$

$$\Rightarrow \cancel{\int_{(S^1\times[0,1],\sigma_{S^1\times[0,1]})} d\gamma^*(d\omega) = \int_{(S^1\times\{1\},\partial} \gamma_1^* = \int_{(S^1\times\{0,1\},\partial\sigma_{S^1\times[0,1]})} \gamma^*\omega \Rightarrow}$$

$$\| $$

$$\cancel{0}$$

$$d\gamma^*\omega = \gamma^* d\omega = 0$$

$$\Rightarrow \int_{(S^1\times[0,1],\sigma_{S^1\times[0,1]})} d\gamma^*\omega = 0$$

$$\int_{\partial(S^1\times[0,1]),\partial\sigma_{S^1\times[0,1]}} \gamma^*\omega = -\int_{(S^1,\sigma)} \gamma_1^*\omega + \int_{(S^1,\sigma)} \gamma_0^*\omega$$

$$\Rightarrow \int_{(S^1,\sigma)} \gamma_1^*\omega = \int_{(S^1,\sigma)} \gamma_0^*\omega$$