Solutions: CalcB, 26 March 2019

Solution to 1. a): $\nabla f = e^{x-y^2}\cos(z)\mathbf{i} - 2ye^{x-y^2}\cos(z)\mathbf{j} - e^{x-y^2}\sin(z)\mathbf{k}$.
b): $\nabla f(0,0,0) = \mathbf{i}$, so the directional derivative is $(1,0,0)\cdot(1,2,3) = 1$.
c): $\frac{\partial^3 f}{\partial x \partial y \partial z} = 2ye^{x-y^2}\sin(z)$.
d): Using the Taylor expansion $e^t = 1+t+t^2/2+o(3)$ and $\cos(z) = 1-z^2+o(3)$, where $o(3)$ denotes terms of order at least three, we obtain that $e^{x-y^2}\cos(z) = (1+x-y^2+(x-y^2)^2/2+o(3))(1-z^2/2+o(3)) = 1+x-y^2+x^2/2-z^2/2+o(3)$. So, the second degree Taylor polynomial is $1+x-y^2+x^2/2-z^2/2$. □

Solution to 2. a): D is the ellipse symmetric wrt the coordinate axes, and which intersects the coordinate axes in $(\pm 2, 0)$ and $(0, \pm 1)$.

b) Since D is closed and bounded and h is a continuous function, h has a global minimum and a global maximum on D. We have that $\nabla h = (1,2) \neq 0$, so h has no critical (stationary) points. This implies that the maximum and the minimum must be on the boundary curve of D, which is given by $g(x,y) = x^2/4 + y^2 - 1 = 0$. Applying the Lagrange multiplier method, we obtain that the minimum and the maximum satisfy the equations $\nabla h(x,y) = \lambda \nabla g(x,y)$ and $g(x,y) = 0$, for some $\lambda \in \mathbb{R}$. The first two equations give $1 = \lambda x/2$, $2 = 2\lambda y$; and so $y = x/2 = 1/\lambda$. Using also the second $x^2/4 + y^2 = 1$, we obtain the following solutions $(x,y) = \pm(\sqrt{2}, 1/\sqrt{2})$. The values of h in these points is $\pm 2\sqrt{2}$. We conclude that $2\sqrt{2}$ is the maximum of f on D and is attained in point $(\sqrt{2}, 1/\sqrt{2})$, and $-2\sqrt{2}$ is the minimum of f on D and is attained in point $(-\sqrt{2}, -1/\sqrt{2})$. □

Solution to 3. a) The level sets of $g(x,y) = e^{x+y^2}$ are the same as the level sets of the function $x + y^2$; thus they are parabolas of the form $x + y^2 = C$; which are the family of translates in the x-direction of the parabola $x = -y^2$.
b) We have that $\nabla g(3,-1) = e^4(\mathbf{i} - 2\mathbf{j})$, so the equation of the tangent line is given by

$$e^4(\mathbf{i} - 2\mathbf{j}) \cdot ((x-3)\mathbf{i} + (y+1)\mathbf{j}) = 0.$$

Equivalently, $x - 2y = 5$.

c) First solution: We have that $\nabla g = e^{x+y^2}(\mathbf{i} + 2y\mathbf{j})$, and so

$$\nabla g \cdot \mathbf{V} = e^{x+y^2}\frac{-2y+2y}{1+x^2} = 0.$$

This shows that $\mathbf{V}$ is perpendicular to ∇g, and since ∇g is normal to the level sets of g, we have that $\mathbf{V}$ is tangent to the level sets. Hence, the level sets are precisely the field lines of $\mathbf{V}$.

Second solution: We have that the field lines of $\mathbf{V}$ satisfy the differential equation $\mathrm{d}x/(-2y) = \mathrm{d}y$, which is equivalent to $\mathrm{d}x + 2y\mathrm{d}y = 0$, hence $\mathrm{d}(x+y^2) = 0$; thus $x + y^2$ is constant on the field lines, and clearly the function $x + y^2$ has the same level sets as g. □

Solution to 4. In spherical coordinates, G is given by

$$0 \leq \cos(\phi) \leq R \leq 2.$$

The first inequality gives $0 \leq \phi \leq \pi/2$. Using $\mathrm{d}x\mathrm{d}y\mathrm{d}z = R^2\sin(\phi)\mathrm{d}\theta\mathrm{d}r\mathrm{d}\phi$, we calculate:

$$\begin{aligned}\iiint_G \frac{\mathrm{d}x\mathrm{d}y\mathrm{d}z}{x^2+y^2+z^2}\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi/2}\mathrm{d}\phi\int_{\cos(\phi)}^2 \sin(\phi)\mathrm{d}R \\ &= 2\pi\int_0^{\pi/2}(2-\cos(\phi))\sin(\phi)\mathrm{d}\phi \\ &= 2\pi\Big(-2\cos(\phi) + \frac{\cos^2(\phi)}{2}\Big)_{\phi=0}^{\phi=\pi/2} \\ &= 3\pi.\end{aligned}$$

□

Solution to 5. b): We have that

$$\mathrm{curl}(\mathbf{F}) = -2ye^z\mathbf{i} - 2xe^z\mathbf{i} + \Big(\cos\Big(\frac{x}{x+y}\Big)\frac{y}{(x+y)^2} - \sin\Big(\frac{x}{x+y}\Big)\frac{x}{(x+y)^2}\Big)\mathbf{k}.$$

c): In cylindrical coordinates, $\mathcal{S}$ is given by

$$r = 1, \quad 0 \leq \theta \leq \pi/2, \quad 0 \leq z \leq \sin^2(\theta).$$

Therefore, a parameterization for $\mathcal{S}$ is given by

$$\mathbf{r}(z, \theta) = (\cos(\theta), \sin(\theta), z), \quad \mathbf{r} : R \to \mathcal{S},$$

where the domain R is described by $0 \leq \theta \leq \pi/2$, $0 \leq z \leq \sin^2(\theta)$. In these coordinates, we have that

$$\frac{\partial \mathbf{r}}{\partial z} \times \frac{\partial \mathbf{r}}{\partial \theta} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ -\sin(\theta) & \cos(\theta) & 0 \end{vmatrix} = -\cos(\theta)\mathbf{i} - \sin(\theta)\mathbf{j}.$$

Note that $\frac{\partial \mathbf{r}}{\partial z} \times \frac{\partial \mathbf{r}}{\partial \theta}$ points towards the positive side of the cylinder (this is why we took z as the first coordinate and θ as the second coordinate), therefore $\mathbf{r}$ is an oriented parameterization. We obtain

$$\mathrm{d}\mathbf{S} = -\big(\cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j}\big)\mathrm{d}z\mathrm{d}\theta.$$

d): We use Stokes' Theorem:

$$\begin{aligned} \oint_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \iint_{\mathcal{S}} \mathbf{curl}(\mathbf{F}) \cdot \mathrm{d}\mathbf{S} \\ &= \int_0^{\pi/2} \mathrm{d}\theta \int_0^{\sin^2(\theta)} 4e^z \sin(\theta)\cos(\theta)\mathrm{d}z \\ &= \int_0^{\pi/2} 4(e^{\sin^2(\theta)} - 1)\sin(\theta)\cos(\theta)\mathrm{d}\theta \\ &= 2\Big(e^{\sin^2(\theta)} - \sin^2(\theta)\Big)_{\theta=0}^{\theta=\pi/2} \\ &= 2(e-2). \end{aligned}$$

□