EXAM: DIFFERENTIAL GEOMETRY
JUNE 18TH 2021, 13:30-16:30

Problem A. *(1 pts)* Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. Let $X \in \mathfrak{X}(M)$ be a vector field on M such that ϕ_X^t, the flow of X, is by local isometries of (M, g), i.e., $(\phi_X^t)^* g = g$.

(1) Show that X satisfies the equation:
$$X\langle Y, Z\rangle = \langle [X, Y], Z\rangle + \langle Y, [X, Z]\rangle, \quad \forall\, Y, Z \in \mathfrak{X}(M).$$
(2) Show that X satisfies the equation from (1) if and only if it satisfies:
$$\langle \nabla_Y X, Z\rangle + \langle Y, \nabla_Z X\rangle = 0, \quad \forall\, Y, Z \in \mathfrak{X}(M).$$

Problem B. *(2.5 pts)* Let $s \in C^\infty(\mathbb{R})$ be a positive function, and consider the following metric on $\mathbb{R}^2$:
$$g = (\mathrm{d}x)^2 + s^2(x)(\mathrm{d}y)^2.$$
(1) Determine the Christoffel symbols of g.
(2) Write the differential equations of geodesics on $(\mathbb{R}^2, g)$. Show that the horizontal lines
$$x(t) = x_0 + pt \quad \text{and} \quad y(t) = y_0,$$
with $x_0, y_0, p \in \mathbb{R}$, are geodesics.
(3) Calculate the parallel transport map along the geodesic segment $\gamma(t) = (t, 0)$ for $t \in [0, 1]$:
$$P_\gamma : T_{(0,0)}\mathbb{R}^2 \to T_{(1,0)}\mathbb{R}^2.$$
(4) Calculate the coefficient R_{1212} of the Riemannian curvature tensor.
(5) Show that the Gaussian curvature is given by $K(x, y) = -\frac{s''(x)}{s(x)}$.
(6) Assume that s is a convex function, i.e., $s''(x) \geq 0$. Show that there are no closed geodesic on $(\mathbb{R}^2, g)$, i.e., there is no geodesic $\gamma : [a, b] \to \mathbb{R}^2$ such that $\gamma(a) = \gamma(b)$ and $\gamma|_{[a,b)}$ is injective.

Problem C. *(2 pts)* Let (M, g) be a *connected* Riemannian manifold without boundary. Recall that (M, g) is called *complete* if M endowed with the Riemannian distance function is a complete metric space. Which of the following implications hold true (either provide a proof, or a precise reference)?
(1) If (M, g) is complete, then for every $p \in M$ the exponential map $\exp_p : T_pM \to M$ is surjective.
(2) If there exists a point $p \in M$ such that $\exp_p$ is defined on T_pM, then M is complete.
(3) If every two points in M can be connected by a minimizing geodesic, then M is complete.
(4) If M is compact then every geodesic γ on M is closed, i.e., there exists $T > 0$ such that $\gamma(0) = \gamma(T)$.

Problem D. *(1.5 pts)* Let (M, g) and (N, h) be two Riemannian manifolds, and let $f \in C^\infty(N)$ be a positive smooth function. Consider the Riemannian manifold:
$$(\widetilde{M}, \widetilde{g}), \quad \widetilde{M} := M \times N, \quad \widetilde{g} := \pi_N^*(f)\pi_M^*(g) + \pi_N^*(h),$$
where $\pi_M : \widetilde{M} \to M$ and $\pi_N : \widetilde{M} \to N$ are the projections. Denote the Levi-Civita connections of the metrics $\widetilde{g}$, g and h by $\widetilde{\nabla}$, ∇^M and ∇^N, respectively.

Recall that the *gradient* of $f \in C^\infty(N)$ is the vector field $\nabla^N f \in \mathfrak{X}(N)$ determined by the relation:
$$Y(f) = h(\nabla^N f, Y), \quad \forall\, Y \in \mathfrak{X}(N).$$
We regard any vector field $X \in \mathfrak{X}(M)$ as the vector field on $\widetilde{M}$, which at $(p, q) \in \widetilde{M}$ is given by:
$$X_p \oplus 0_q \in T_pM \oplus T_qN = T_{(p,q)}\widetilde{M}.$$
Similarly, vector fields on N we view also as vector fields on $\widetilde{M}$.

(1) Prove that, for all $X_1, X_2 \in \mathfrak{X}(M)$ and all $Y_1, Y_2 \in \mathfrak{X}(N)$, the following identities hold:
$$\widetilde{\nabla}_{X_1} X_2 = \nabla^M_{X_1} X_2 - \frac{1}{2} g(X_1, X_2)\nabla^N f,$$
$$\widetilde{\nabla}_{X_1} Y_1 = \frac{1}{2} Y_1(\log(f))X_1,$$
$$\widetilde{\nabla}_{Y_1} X_1 = \frac{1}{2} Y_1(\log(f))X_1,$$
$$\widetilde{\nabla}_{Y_1} Y_2 = \nabla^N_{Y_1} Y_2.$$
(2) For a fixed $q \in N$, calculate the second fundamental of the submanifold $M \times \{q\}$ of $\widetilde{M}$.