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Exam for the course “Category Theory and Homological Algebra”

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- Please write your name and your student registration number on each page.
- Questions may be answered in Dutch or English
- It is not allowed to use any notes, books or electronic devices during the exam.
- Only solutions on exam paper will be graded.
- Please write readably and justify your answers.
- Read through all problems before you start to work.
- If you are unable to solve one of the items of a question, you are nonetheless allowed (and encouraged!) to use the results from this item for your solutions of the subsequent items.
- There are $30 + 25 + 15 = 70$ points in total.
- **Students taking the bachelor variant of the course should skip the last items of exercises 1 and 2, that is, 1(vi) and 2(v)!**

Problem 1. (30 = 5+5+5+5+5+5 points)

In this exercise, all modules are right R-modules over a not necessarily commutative ring R. We say that an R-module E is *injective* if for every injective R-module homomorphism $i\colon M \to N$ and every R-module homomorphism $f\colon M \to E$, there exists an R-module homomorphism $g\colon N \to E$ with $g \circ i = f$. In other words, for every solid arrow diagram

$$\begin{array}{ccccc} & & E & & \\ & & f\uparrow & \nwarrow g & \\ 0 & \to & M & \xrightarrow[i]{} & N \end{array}$$

in which the bottom line is exact there exists a dotted arrow g making the triangle commutative.

(i) Show that an R-module E is injective if and only if the functor $\mathrm{Hom}_R(-, E)\colon (\mathsf{Mod}_R)^{\mathrm{op}} \to \mathsf{Ab}$ is exact.

(ii) Let $(E_\alpha)_{\alpha \in A}$ be a family of R-modules indexed by a set A. Show that the product $\prod_{\alpha \in A} E_\alpha$ is injective if and only if each of the E_α is injective.

(iii) Show that if E is injective, then every short exact sequence

$$0 \longrightarrow E \xrightarrow{h} U \xrightarrow{k} V \longrightarrow 0$$

of R-modules splits.

(iv) Show that if E is an R-module such that every short exact sequence

$$0 \longrightarrow E \xrightarrow{h} U \xrightarrow{k} V \longrightarrow 0$$

of R-modules splits, then E is injective.

(v) Suppose in addition that R is an integral domain, that is, a commutative ring without zero divisors. Show that if E is an injective R-module, then for every $e \in E$ and every $0 \neq r \in R$ there exists an $e' \in E$ with $e = e'r$.

(vi) Suppose again that R is an integral domain. Show that if there exists a non-zero R-module E that is both injective and projective, then R is a field. (Hint: First use (v) to show that every non-zero homomorphism $g\colon E \to R$ has $1 \in R$ in its image.)

Solution. (i) The definition can be rephrased by saying that E is injective if and only if for any injection $i\colon M \to N$, the induced map $i^* = \operatorname{Hom}_R(i, E)\colon \operatorname{Hom}_R(N, E) \to \operatorname{Hom}_R(M, E)$ is surjective. Since we know that $\operatorname{Hom}_R(-, E)$ is left exact, the claim follows.

(ii) Suppose the product is injective and we are given an injection $i\colon M \to N$ and a homomorphism $f_\alpha\colon M \to E_\alpha$. Applying the universal property of the product to f_α and the zero maps into the other factor provides a homomorphism $f\colon M \to \prod_{\alpha\in A} E_\alpha$. Extending it over i and composing the resulting extension with the projection to E_α gives the desired extension of f_α.

For the other implication, we are given an injection $i\colon M \to N$ and a homomorphism $f\colon M \to \prod_{\alpha\in A} E_\alpha$. Composing the latter with the projections to the E_α, extending the composites over i and using the universal property of the product to combine these extensions to a homomorphism $g\colon N \to \prod_{\alpha\in A} E_\alpha$ shows the injectivity of the product.

(iii) Applying the definition to the injection $h\colon E \to U$ and $\mathrm{id}_E\colon E \to E$ provides a retraction $\rho\colon U \to E$, i.e., an R-module homomorphism ρ with $\rho h = \mathrm{id}_E$. This shows the splitting.

(iv) Given $i\colon M \to N$ and $f\colon M \to E$ as in the definition of injectivity, we take the pushout of the diagram given by these two maps and obtain the following commutative diagram where the lines are exact:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E & \xrightarrow{j} & E \oplus_M N & \xrightarrow{q} & \operatorname{coker}(j) & \longrightarrow & 0 \\
 & & {\scriptstyle f}\uparrow & & {\scriptstyle g}\uparrow & & \uparrow & & \\
0 & \longrightarrow & M & \xrightarrow{i} & N & \xrightarrow{q} & \operatorname{coker}(i) & \longrightarrow & 0
\end{array}$$

Then (iii) provides a retraction ρ for j, and the composite ρg is the desired extension of f since $\rho g i = \rho j f = f$.

Alternatively, one can use isomorphism between the Ext-groups defined via projective resolutions and the Yoneda-Ext groups to see that the splitting condition implies the vanishing of $\operatorname{Ext}^1_R(-, E)$. Applying the long exac sequence for Ext, it follows that $\operatorname{Hom}_R(-, E)$ is exact and that E is injective by (i).

(v) For r and e as in the problem, we consider the R-module $M = \{rr' \mid r' \in R\}$ and the R-module homomorphism $f\colon M \to E$ defined by $f(rr') = er'$. It is well-defined since R has no zero divisors. Injectivity of E implies that f extends over the inclusion $i\colon M \to R$, giving an R-module homomorphism $g\colon R \to E$ with $gi = f$. Thus $e = f(r) = g(r) = g(1)r$.

(vi) Since E is projective, there exist a set A an injection $f\colon E \to \oplus_{\alpha\in A} R$. Because E is non-zero, one of the projections $p_\beta\colon \oplus_{\alpha\in A} R \to R$ has the property $p_\beta \circ f \neq 0$. Writing φ_r for the left multiplication with $0 \neq r \in R$, the assumption that R is a domain implies that $g = \varphi_r \circ p_\beta \circ f \neq 0$. For an $e \in E$ with $f(e) \neq 0$, we can use (v) to find an $e' \in E$ with $e = e'g(e)$ and thus $g(e) = g(e')g(e)$. This implies $1 = g(e') = r(p_\beta \circ f)(e')$. Hence r is invertible.

Problem 2. (25 = 5+5+5+5+5 points)

Let R and S be (not necessarily commutative) rings and let Mod_R and Mod_S be the associated categories of right modules. We consider a functor

$$F\colon \mathsf{Mod}_R \to \mathsf{Mod}_S$$

with the following three properties:

(a) The functor F is right exact.

(b) The functor F preserves direct sums, that is, the canonical map $F(M) \oplus F(M') \to F(M \oplus M')$ is an isomorphism for all right R-modules M and M'.

(c) The functor F is additive on morphism sets, that is, for all right R-modules M and M' the map $\operatorname{Hom}_R(M, M') \to \operatorname{Hom}_R(F(M), F(M')), \varphi \mapsto F(\varphi)$ is a homomorphism of abelian groups.

We know from the lecture that if N is an (R,S)-bimodule, then $- \otimes_R N\colon \mathsf{Mod}_R \to \mathsf{Mod}_S$ has properties (a)-(c). The aim of this problem is to show that every functor F satisfying (a)-(c) is isomorphic to the tensor product with an (R,S)-bimodule. This will be done in several steps:

(i) Let M be an R-module and let $m \in M$ be an element of M. Consider the map

$$\varphi_m\colon R \to M, \qquad r \mapsto mr \ .$$

Show that φ_m is a morphism of right R-modules.

(ii) For $r \in R$, we write $\varphi_r\colon R \to R$ for the morphism obtained from (i) when setting $M = R$. Show that $r \cdot x = F(\varphi_r)(x)$ defines a left R-module structure on $F(R)$ that turns $F(R)$ into an (R,S)-bimodule.

(iii) Let M be a right R-module. Show that the map

$$M \times F(R) \to F(M), \qquad (m,x) \mapsto F(\varphi_m)(x)$$

induces a homomorphism $\sigma_M\colon M \otimes_R F(R) \to F(M)$ of right S-modules.

(iv) Show that the homomorphisms σ_M from (iii) assemble to a natural transformation

$$\sigma\colon -\otimes_R F(R) \to F$$

of functors $\mathsf{Mod}_R \to \mathsf{Mod}_S$.

(v) Show that the natural transformation σ from (iv) is a natural isomorphism.
(Hint: Start with the case $M = R$!)

Solution. (i) We have

$$\varphi_m(r_1 r_2) = m(r_1 r_2) = (mr_1)r_2 = \varphi_m(r_1)r_2$$

and

$$\varphi(r_1 + r_2) = m(r_1 + r_2) = mr_1 + mr_2 = \varphi_m(r_1) + \varphi_m(r_2)$$

for all $r_1, r_2 \in R$ and $m \in M$.

(ii) For all $x, x_1, x_2 \in F(R)$ and $r, r_1, r_2 \in R$ we have

$$1 \cdot x = F(\varphi_1)(x) = F(\mathrm{id}_R)(x) = \mathrm{id}_{F(R)}(x) = x,$$

$$r(x_1 + x_2) = F(\varphi_r)(x_1 + x_2) = F(\varphi_r)(x_1) + F(\varphi_r)(x_2) = rx_1 + rx_2$$

since $F(\varphi_r)$ is homomorphism of right S-modules, and

$$(r_1 r_2)x = F(\varphi_{r_1 r_2})(x) = F(\varphi_{r_1} \circ \varphi_{r_2})(x) = F(\varphi_{r_1})(F(\varphi_{r_2})(x)) = r_1(r_2 x).$$

The last part of the R-module structure

$$(r_1 + r_2)x = F(\varphi_{r_1 + r_2})(x) = F(\varphi_{r_1} + \varphi_{r_2})(x) = (F(\varphi_{r_1}) + F(\varphi_{r_2}))(x) = r_1 x + r_2 x$$

uses property (c).

To see that this defines a bimodule structure, we notice that for $s \in S$ we have

$$r(xs) = F(\varphi_r)(xs) = F(\varphi_r)(x)s = (rx)s$$

since $F(\varphi_r)$ is homomorphism of right S-modules.

(iii) The given map is R-bilinear since for all $m, m_1, m_2 \in M$, $x, x_1, x_2 \in F(R)$, and $r \in R$, we have

$$F(\varphi_{m_1+m_2})(x) = F(\varphi_{m_1})(x) + F(\varphi_{m_2})(x), \quad F(\varphi_m)(x_1 + x_2) = F(\varphi_m)(x_1) + F(\varphi_m)(x_2)$$

and

$$F(\varphi_{mr})(x) = F(\varphi_m \circ \varphi_r)(x) = F(\varphi_m)(rx).$$

So the universal property of the tensor product implies that there is an induced homomorphism $\sigma_M\colon M \otimes_R F(R) \to F(M)$ of abelian groups. It is a homomorphism of S-modules because on generators of the form $m \otimes x$, we have

$$\sigma_M((m \otimes x)s) = \sigma_M((m \otimes (xs)) = F(\varphi_m)(xs) = F(\varphi_m(x))s = \sigma(m \otimes x)s.$$

(iv) Let $f\colon M \to N$ be a homomorphism of right R-modules. Then

$$\begin{aligned}(F(f) \circ \sigma_M)(m \otimes x) = F(f)(F(\varphi_m)(x)) = F(f \circ \varphi_m)(x) = F(\varphi(f(m)))(x) \\ = \sigma_N(f(m) \otimes x) = (\sigma_N \circ (f \otimes_R F(R)))(x)\end{aligned}$$

holds for all $m \in M$ and $x \in F(R)$.

(v) The canonical isomorphism $R \otimes_R F(R) \to F(R)$ shows that σ_M is an isomorphism if $M = R$. Since the tensor product $- \otimes_R F(R)$ preserves direct sums, it follows that σ_M is an isomorphism for any free R-module M. For the general case, we choose an exact sequence $P_1 \to P_0 \to M \to 0$ with P_1, P_0 free R-modules and apply the five lemma to the diagram resulting from applying the natural transformation to this sequence.

Problem 3. (15 = 5+5+5 points)

Let C be the subcategory of the category of sets whose objects are finite sets and whose morphisms are bijections between sets. For a finite set X with n elements, we consider the set

$$\mathrm{Perm}(X) = \{f\colon X \to X \mid f \text{ is a bijective map of sets}\}$$

of permutations of X and the set

$$\mathrm{Ord}(X) = \{f\colon \{1, \dots, n\} \to X \mid f \text{ is a bijective map of sets}\}$$

of total orderings on X. (A bijection f as considered in the definition of $\mathrm{Ord}(X)$ may be viewed as ordering of X by defining $f(1) < \dots < f(n)$, and it is clear that every possible ordering corresponds to such a bijection.)

(i) Show that these definitions extend to covariant functors $\mathrm{Perm}\colon \mathsf{C} \to \mathsf{Set}$ and $\mathrm{Ord}\colon \mathsf{C} \to \mathsf{Set}$.
(ii) Show that there is a bijection $\mathrm{Perm}(X) \to \mathrm{Ord}(X)$ for every finite set X.
(iii) Show that there is no natural transformation $\mathrm{Perm} \to \mathrm{Ord}$.

Solution. (i) For a bijection $g\colon X \to Y$, we define

$$\mathrm{Perm}(g)\colon \mathrm{Perm}(X) \to \mathrm{Perm}(Y), \qquad f \mapsto gfg^{-1}$$

and

$$\mathrm{Ord}(g)\colon \mathrm{Ord}(X) \to \mathrm{Ord}(Y), \qquad f \mapsto gf$$

It is clear that $\mathrm{Perm}(\mathrm{id}_X) = \mathrm{id}_{\mathrm{Perm}(X)}$ and $\mathrm{Ord}(\mathrm{id}_X) = \mathrm{id}_{\mathrm{Ord}(X)}$. If $h\colon Y \to Z$ is another bijection, we also have $h(gfg^{-1})h^{-1} = (hg)f(hg)^{-1}$ and $h(gf) = (hg)f$. This shows functoriality.

(ii) If X has n elements, then both $\mathrm{Perm}(X)$ and $\mathrm{Ord}(X)$ have $n!$ elements. Thus there is a bijection. Alternatively, the choice of a bijection $k\colon X \to \{1, \dots, n\}$ induces a bijection $k_*\colon \mathrm{Ord}(X) \to \mathrm{Perm}(X)$.

(iii) Suppose τ is such a natural transformation. Consider the set $X = \{a, b\}$ and the non-identity bijection $f\colon X \to X$. Then $\mathrm{Perm}(g) = \mathrm{id}_X$. Naturality of τ therefore implies

$$\mathrm{Ord}(g)(\tau_X(\mathrm{id}_X)) = \tau_X((\mathrm{Perm}(g))(\mathrm{id}_X)) = \tau_X(\mathrm{id}_X).$$

Since $\mathrm{Ord}(g)(f) = f$ implies $g = \mathrm{id}_X$, this is a contradiction.