Solution to 1. The given inequalities describing D imply that $x, y, z \geq 0$, and so $x^2 \leq z^2$ implies $x \leq z$. Note also that $x^2 \leq x$ implies that $x \in [0,1]$. Therefore, we can give the equivalent description of D:

$$0 \leq x \leq 1, \quad x \leq z \leq \sqrt{x}, \quad x^2 \leq y \leq z^2.$$

We calculate:

$$\begin{aligned}\iiint_D \frac{20}{z-x} \mathrm{d}x\mathrm{d}y\mathrm{d}z &= \int_0^1 \mathrm{d}x \int_x^{\sqrt{x}} \mathrm{d}z \int_{x^2}^{z^2} \frac{20}{z-x}\ \mathrm{d}y = \\ &= \int_0^1 \mathrm{d}x \int_x^{\sqrt{x}} \frac{20(z^2-x^2)}{z-x}\ \mathrm{d}z = \\ &= \int_0^1 \mathrm{d}x \int_x^{\sqrt{x}} 20(z+x)\ \mathrm{d}z = \\ &= \int_0^1 20(z^2/2 + xz)\Big|_{z=x}^{z=\sqrt{x}}\ \mathrm{d}x = \\ &= \int_0^1 20(x/2 - x^2/2 + x^{3/2} - x^2)\ \mathrm{d}x = \\ &= \int_0^1 10(x - 3x^2 + 2x^{3/2})\ \mathrm{d}x = \\ &= 10(1/2 - 1 + 4/5) = \\ &= 5 - 10 + 8 = 3.\end{aligned}$$

□

Solution to 2. b): $\mathrm{d}x\mathrm{d}y = 2u\mathrm{d}u\mathrm{d}v$.
c): The domain B is the quarter-disk $u^2 + v^2 \leq 1$, $0 \leq u$, $0 \leq v$. Using the above change of coordinates, and then using polar coordinates $u = r\cos(\theta)$, $v = r\sin(\theta)$, we calculate the integral:

$$\begin{aligned}\iint_A \sqrt{y-x}\, e^{(y+x^2-x)^2} \mathrm{d}x\mathrm{d}y &= \iint_B u e^{(u^2+v^2)^2} 2u\mathrm{d}u\mathrm{d}v = \\ &= \int_0^{\pi/2} 2\cos^2(\theta)\mathrm{d}\theta \int_0^1 r^3 e^{r^4}\mathrm{d}r = \\ &= \int_0^{\pi/2} (\cos(2\theta)+1)\mathrm{d}\theta \int_0^1 r^3 e^{r^4}\mathrm{d}r = \\ &= \left(\frac{\sin(2\theta)}{2} + \theta\right)\Big|_{\theta=0}^{\theta=\pi/2} \frac{e^{r^4}}{4}\Big|_{r=0}^{r=1} = \\ &= \frac{\pi(e-1)}{8}.\end{aligned}$$

□

Solution to 3. b): Using spherical coordinates, we obtain the following parametrization for the surface $\mathcal{S}$ of D:

$$\mathbf{r}(\phi,\theta) = (\sin(\phi)^2\cos(\theta), \sin(\phi)^2\sin(\theta), \sin(\phi)\cos(\phi)),$$

with $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$. Therefore,

$$\mathrm{d}\mathbf{S} = \frac{\partial\mathbf{r}}{\partial\phi} \times \frac{\partial\mathbf{r}}{\partial\theta}\mathrm{d}\phi\mathrm{d}\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\sin(\phi)\cos(\phi)\cos(\theta) & 2\sin(\phi)\cos(\phi)\sin(\theta) & \cos^2(\phi)-\sin^2(\phi) \\ -\sin^2(\phi)\sin(\theta) & \sin^2(\phi)\cos(\theta) & 0 \end{vmatrix} \mathrm{d}\phi\mathrm{d}\theta.$$

Note that this parameterization is such that the resulting normal vector field points out of $\mathcal{S}$. Namely, by calculating $\mathrm{d}\mathbf{S}$ at $\mathbf{r}(\phi = \pi/2, \theta = 0) = (1,0,0)$ we obtain

$$\mathrm{d}\mathbf{S} = \frac{\partial\mathbf{r}}{\partial\phi} \times \frac{\partial\mathbf{r}}{\partial\theta}\mathrm{d}\phi\mathrm{d}\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix} \mathrm{d}\phi\mathrm{d}\theta = \mathbf{i}\ \mathrm{d}\phi\mathrm{d}\theta,$$

and so the normal vector field is $\mathbf{N}(1,0,0) = \mathbf{i}$.

On $\mathcal{S}$, in the same parametrization, we have that

$$\mathbf{F} = \sin(\phi)^2\cos(\theta)\mathbf{i} + \sin(\phi)^2\sin(\theta)\mathbf{j} + \sin(\phi)\cos(\phi)\mathbf{j},$$

and so

$$\mathbf{F}\cdot \mathrm{d}\mathbf{S} = \begin{vmatrix} \sin(\phi)^2\cos(\theta) & \sin(\phi)^2\sin(\theta) & \sin(\phi)\cos(\phi) \\ 2\sin(\phi)\cos(\phi)\cos(\theta) & 2\sin(\phi)\cos(\phi)\sin(\theta) & \cos^2(\phi)-\sin^2(\phi) \\ -\sin^2(\phi)\sin(\theta) & \sin^2(\phi)\cos(\theta) & 0 \end{vmatrix}\mathrm{d}\phi\mathrm{d}\theta =$$

$$= \sin^3(\phi)\begin{vmatrix} \sin(\phi)\cos(\theta) & \sin(\phi)\sin(\theta) & \cos(\phi) \\ -\sin(\theta) & \cos(\theta) & 0 \\ 2\sin(\phi)\cos(\phi)\cos(\theta) & 2\sin(\phi)\cos(\phi)\sin(\theta) & \cos^2(\phi)-\sin^2(\phi) \end{vmatrix}\mathrm{d}\phi\mathrm{d}\theta =$$

$$= \sin^3(\phi)\begin{vmatrix} \sin(\phi)\cos(\theta) & \sin(\phi)\sin(\theta) & \cos(\phi) \\ \sin(2\phi)\cos(\theta) & \sin(2\phi)\sin(\theta) & \cos(2\phi) \\ -\sin(\theta) & \cos(\theta) & 0 \end{vmatrix}\mathrm{d}\phi\mathrm{d}\theta =$$

$$= -\sin^3(\phi)\big(\sin(\phi)\cos(2\phi)\cos^2(\theta) - \cos(\phi)\sin(2\phi)\sin^2(\theta)$$
$$+ \sin(\phi)\cos(2\phi)\sin^2(\theta) - \cos(\phi)\sin(2\phi)\cos^2(\theta)\big)\mathrm{d}\phi\mathrm{d}\theta =$$
$$= -\sin^3(\phi)\big(\sin(\phi)\cos(2\phi) - \cos(\phi)\sin(2\phi)\big)\mathrm{d}\phi\mathrm{d}\theta =$$
$$= -\sin^3(\phi)\big(\sin(\phi)(\cos^2(\phi)-\sin^2(\phi)) - \cos(\phi)2\sin(\phi)\cos(\phi)\big)\mathrm{d}\phi\mathrm{d}\theta =$$
$$= \sin^4(\phi)\mathrm{d}\phi\mathrm{d}\theta.$$

Thus, we can now calculate the flux of $\mathbf{F}$ out of $\mathcal{S}$, and obtain:

$$\iint_{\mathcal{S}} \mathbf{F}\cdot\mathrm{d}\mathbf{S} = \int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\sin^4(\phi)\mathrm{d}\phi =$$
$$= 2\pi\int_0^{\pi}\frac{1}{4}(1-\cos(2\phi))^2\mathrm{d}\phi =$$
$$= \frac{\pi}{2}\int_0^{\pi}(1-2\cos(2\phi)+\cos^2(2\phi))\mathrm{d}\phi =$$
$$= \frac{\pi}{2}\int_0^{\pi}(1-2\cos(2\phi)+\frac{1}{2}(1+\cos(4\phi)))\mathrm{d}\phi =$$
$$= \frac{\pi}{2}\Big(\frac{3\pi}{2} - \sin(2\phi)\big|_{\phi=0}^{\phi=\pi} + \frac{1}{8}\sin(4\phi)\big|_{\phi=0}^{\phi=\pi}\Big) =$$
$$= \frac{3\pi^2}{4}.$$

We have that $\mathrm{div}(\mathbf{F}) = 3$. Using again spherical coordinates, we obtain:

$$\iiint_D \mathbf{div}(\mathbf{F})\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\mathrm{d}\phi\int_0^{\sin(\phi)} 3R^2\sin(\phi)\mathrm{d}R =$$
$$= 2\pi\int_0^{\pi}\sin(\phi)^4\mathrm{d}\phi.$$

We obtain the same integral as above, therefore we obtain the relation in the divergence theorem:

$$\iint_{\mathcal{S}} \mathbf{F}\cdot\mathrm{d}\mathbf{S} = \iiint_D \mathbf{div}(\mathbf{F})\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$ □

Solution to 4. b): $\mathbf{curl}(\mathbf{F}) = 2x\mathbf{i} - 2z\mathbf{k}$.

c): We use cylindrical coordinates to parameterize $\mathcal{S}$:

$$\mathbf{r}(\theta,z) = (\cos(\theta),\sin(\theta),z),\quad 0\le\theta\le 2\pi,\ \cos(\theta)\le z\le 2.$$

We have that

$$\mathrm{d}\mathbf{S} = \frac{\partial\mathbf{r}}{\partial\theta}\times\frac{\partial\mathbf{r}}{\partial z}\mathrm{d}\theta\mathrm{d}z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix}\mathrm{d}\theta\mathrm{d}z = (\cos(\theta)\mathbf{i}+\sin(\theta)\mathbf{j})\mathrm{d}\theta\mathrm{d}z.$$

Note that this with this choice, the normal vector field points out of $\mathcal{S}$.

In these coordinates, on $\mathcal{S}$, we have that

$$\mathbf{curl}(\mathbf{F}) = 2\cos(\theta)\mathbf{i} - 2z\mathbf{k}.$$

Therefore

$$\iint_{\mathcal{S}} \mathbf{curl}(\mathbf{F}) \cdot \mathrm{d}\mathbf{S} = \int_0^{2\pi} \mathrm{d}\theta \int_{\cos(\theta)}^{2} 2\cos(\theta)^2 \mathrm{d}z =$$

$$= 2\int_0^{2\pi} (2-\cos(\theta))\cos(\theta)^2 \mathrm{d}\theta =$$

$$= 2\int_0^{2\pi} (1+\cos(2\theta) - \cos(\theta)(1-\sin(\theta)^2))\mathrm{d}\theta =$$

$$= 4\pi + 2\big(\frac{1}{2}\sin(2\theta) - \sin(\theta) + \frac{1}{3}\sin(\theta)^3\big)\Big|_{\theta=0}^{\theta=2\pi} =$$

$$= 4\pi.$$

d): An oriented parameterization of $\mathcal{C}_1$ is given by:

$$\mathbf{r}(\theta) = (\cos(\theta), -\sin(\theta), 1).$$

Therefore:

$$\oint_{\mathcal{C}_1} yz \ \mathrm{d}x - xz \ \mathrm{d}y + xy \ \mathrm{d}z = \int_0^{2\pi} 2(\sin(\theta)^2 + \cos(\theta)^2)\mathrm{d}\theta = 4\pi$$

e): By Stokes' Theorem:

$$\oint_{\mathcal{C}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_{\mathcal{S}} \mathbf{curl}(\mathbf{F}) \cdot \mathrm{d}\mathbf{S} - \oint_{\mathcal{C}_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0.$$

For verification, let us perform the calculation also directly. An oriented parameterization of $\mathcal{C}_2$ is given by:

$$\mathbf{r}(\theta) = (\cos(\theta), \sin(\theta), \cos(\theta)).$$

Therefore:

$$\oint_{\mathcal{C}_2} yz \ \mathrm{d}x - xz \ \mathrm{d}y + xy \ \mathrm{d}z = -\int_0^{2\pi} (\cos(\theta)\sin(\theta)^2 + \cos(\theta)^3 + \cos(\theta)\sin^2(\theta))\mathrm{d}\theta =$$

$$= -\int_0^{2\pi} \cos(\theta)(1+\sin^2(\theta))\mathrm{d}\theta =$$

$$= -\int_0^{2\pi} \sin(\theta) + \frac{1}{3}\sin^3(\theta))\mathrm{d}\theta = 0.$$

□