Solution to 1. Consider D' the domain given by the inequalities:

$$0 \leq x, y, z \leq 1, \quad x + y + z \leq 1.$$

Then $[0,1]^3 = D \cup D'$, and D and D' overlap only on the boundary. Thus:

$$\iiint_{[0,1]^3} y \mathrm{d}x \mathrm{d}y \mathrm{d}z = \iiint_D y \mathrm{d}x \mathrm{d}y \mathrm{d}z + \iiint_{D'} y \mathrm{d}x \mathrm{d}y \mathrm{d}z.$$

We have that

$$\iiint_{[0,1]^3} y \mathrm{d}x \mathrm{d}y \mathrm{d}z = \int_0^1 \mathrm{d}x \int_0^1 y \mathrm{d}y \int_0^1 \mathrm{d}z = \frac{1}{2}.$$

On the other hand:

$$\iiint_{D'} y \mathrm{d}x \mathrm{d}y \mathrm{d}z = \int_0^1 y \mathrm{d}y \int_0^{1-y} \mathrm{d}x \int_0^{1-x-y} \mathrm{d}z = \int_0^1 y \mathrm{d}y \int_0^{1-y} (1-x-y) \mathrm{d}x =$$
$$= \int_0^1 y \left((1-y)^2 - \frac{(1-y)^2}{2} \right) \mathrm{d}y = \frac{1}{2} \int_0^1 \left(y - 2y^2 + y^3 \right) \mathrm{d}y = \frac{1}{4} - \frac{1}{3} + \frac{1}{8} = \frac{6-8+3}{24} = \frac{1}{24}.$$

Therefore,

$$\iiint_D y \mathrm{d}x \mathrm{d}y \mathrm{d}z = \frac{1}{2} - \frac{1}{24} = \frac{11}{24}.$$

□

Solution to 2. b): $\mathrm{d}x\mathrm{d}y = |u \sin(t)| \mathrm{d}u\mathrm{d}t = u \sin(t) \mathrm{d}u\mathrm{d}t$.

c): Under the given change of variable, the interior of $\mathcal{C}$ corresponds to the points (u, t) so that $t \in [0, \pi]$ and $0 \leq u \leq \sin(t)$. Therefore the area is given by

$$\int_0^\pi \mathrm{d}t \sin(t) \int_0^{\sin(t)} u \mathrm{d}u = \frac{1}{2} \int_0^\pi \sin(t)^3 \mathrm{d}t = \frac{1}{2} \int_0^\pi (1 - \cos(t)^2) \sin(t) \mathrm{d}t =$$
$$= \frac{1}{2} \left[-\cos(t) + \frac{1}{3} \cos(t)^3 \right]_{t=0}^{t=\pi} = \frac{1}{2}(2 - \frac{2}{3}) = \frac{2}{3}.$$

d): Let D denote the domain enclosed by $\mathcal{C}$. Note that the orientation induced by $\mathbf{r}$ of $\mathcal{C}$ is clockwise orientation. We apply Green's theorem for the vector field $\mathbf{F} = y\mathbf{i}$; we obtain:

$$-\iint_D \mathrm{d}x \mathrm{d}y = -\int_{\mathcal{C}} y \mathrm{d}x = -\int_0^\pi \sin(t) \cos(t)^2 \mathrm{d}t = \frac{1}{3} \cos^3(t) \Big|_{t=0}^{t=\pi} = -\frac{2}{3};$$

hence the area of D is $2/3$.

□

Solution to 3. b): The usual formula for a surface given as the graph of a map $z = g(x, y) = e^x \cos(y)$ gives:

$$\mathrm{d}\mathbf{S} = (-\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}) \mathrm{d}x \mathrm{d}y = (-e^x \cos(y)\mathbf{i} + e^x \sin(y)\mathbf{j} + \mathbf{k}) \mathrm{d}x \mathrm{d}y.$$

c):(*first solution*) Consider the vector field:

$$\mathbf{F} = -\sin(y) e^{2x} \sqrt{1 + e^{2x}}\mathbf{i} + \cos(y) e^{2z}\mathbf{j} + 2\sin(y) e^{2z}\mathbf{k}.$$

We calculate:

$$\mathrm{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\sin(y) e^{2x} \sqrt{1 + e^{2x}} & \cos(y) e^{2z} & 2\sin(y) e^{2z} \end{vmatrix} =$$
$$= (2\cos(y) e^{2z} - 2\cos(y) e^{2z})\mathbf{i} - (0 - 0)\mathbf{j} + (0 - \cos(y) e^{2x} \sqrt{1 + e^{2x}})\mathbf{k} =$$
$$= -\cos(y) e^{2x} \sqrt{1 + e^{2x}}\mathbf{k}$$

Applying Stokes' Theorem, we obtain:

$$\begin{aligned}\oint_{\mathcal{C}} \mathbf{F}\cdot \mathrm{d}\mathbf{r} &= \iint_{\mathcal{S}} \mathrm{curl}(\mathbf{F})\cdot \mathrm{d}\mathbf{S} = \\ &= \int_0^1 e^{2x}\sqrt{1+e^{2x}}\mathrm{d}x \int_0^{\pi/2} \cos(y)\mathrm{d}y = \\ &= \frac{1}{3}(1+e^{2x})^{3/2}\Big|_{x=0}^{x=1} \cdot \sin(y)\Big|_{y=0}^{y=\pi/2} = \\ &= \frac{(1+e^2)^{3/2}-2^{3/2}}{3}.\end{aligned}$$

c): (*second solution*). We calculate the integral directly. First note that:

$$\mathbf{F} = -\sin(y)e^{2x}\sqrt{1+e^{2x}}\mathbf{i} + \nabla\Phi,$$

where the function Φ is given by:

$$\Phi(x,y,z) = \sin(y)e^{2z}.$$

Since $\mathcal{C}$ is a closed curve, the circulation of the conservative vector field $\nabla\Phi$ on $\mathcal{C}$ is zero. Hence:

$$\int_{\mathcal{C}} \mathbf{F}\cdot \mathrm{d}r = -\int_{\mathcal{C}} \sin(y)e^{2x}\sqrt{1+e^{2x}}\mathrm{d}x + \int_{\mathcal{C}} \nabla\Phi = -\int_{\mathcal{C}} \sin(y)e^{2x}\sqrt{1+e^{2x}}\mathrm{d}x.$$

Next, decompose $\mathcal{C}$ into its 4 smooth components: $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$, parameterized as follows:

$$\begin{aligned}\mathcal{C}_1 &: \ (x,0,e^x) \ \ 0\le x\le 1\\ \mathcal{C}_2 &: \ (1,y,e\cos(y)) \ \ 0\le y\le \pi/2\\ \mathcal{C}_3 &: \ (x,\pi/2,0) \ \ 0\le x\le 1\\ \mathcal{C}_4 &: \ (0,y,\cos(y)) \ \ 0\le y\le \pi/2.\end{aligned}$$

We have:

$$\begin{aligned}-\int_{\mathcal{C}_1} \sin(y)e^{2x}\sqrt{1+e^{2x}}\mathrm{d}x &= \int_0^1 0\mathrm{d}x = 0;\\ -\int_{\mathcal{C}_2} \sin(y)e^{2x}\sqrt{1+e^{2x}}\mathrm{d}x &= \int_0^{\pi/2} 0\mathrm{d}y = 0;\\ -\int_{\mathcal{C}_3} \sin(y)e^{2x}\sqrt{1+e^{2x}}\mathrm{d}x &= -\int_0^1 e^{2x}\sqrt{1+e^{2x}}\mathrm{d}x = \\ &= -\frac{1}{3}(1+e^{2x})^{3/2}\Big|_{x=0}^{x=1} = \frac{2^{3/2}-(1+e^2)^{3/2}}{3};\\ -\int_{\mathcal{C}_4} \sin(y)e^{2x}\sqrt{1+e^{2x}}\mathrm{d}x &= \int_0^{\pi/2} 0\mathrm{d}y = 0.\end{aligned}$$

Taking into account the orientation, we have that:

$$\int_{\mathcal{C}} = \int_{\mathcal{C}_1} + \int_{\mathcal{C}_2} - \int_{\mathcal{C}_3} - \int_{\mathcal{C}_4};$$

therefore, we recover the answer from the first solution

$$-\int_{\mathcal{C}} \sin(y)e^{2x}\sqrt{1+e^{2x}}\mathrm{d}x = -\frac{2^{3/2}-(1+e^2)^{3/2}}{3}.$$ □

Solution to 4. b)(*first solution*): Using the obvious parameterization of $\mathcal{C}$: $x = a\cos(t)$, $y = a\sin(t)$ and $z = 0$, we obtain:

$$\oint_{\mathcal{C}} z\mathrm{d}x + x\mathrm{d}y + y\mathrm{d}z = \int_0^{2\pi} a^2\cos(t)^2\mathrm{d}t = a^2\pi.$$

b)(*second solution*): Apply Stokes' to $\mathcal{S}_2$, oriented upwards. Then $\mathbf{N} = \mathbf{k}$, and

$$\mathbf{curl}(z\mathbf{i} + x\mathbf{j} + y\mathbf{k}) = \mathbf{i} + \mathbf{j} + \mathbf{k}.$$

Therefore,

$$\oint_{\mathcal{C}} z\mathrm{d}x + x\mathrm{d}y + y\mathrm{d}z = \iint_{\mathcal{S}_2} \mathrm{d}x\mathrm{d}y = a^2\pi.$$

c): Consider the following parameterization of $\mathcal{S}_1$:

$$x = r\cos(t), \quad y = r\sin(t), \quad z = \sqrt{a^2 - r^2}, \quad 0 \le t \le 2\pi,\ 0 \le r \le a.$$

Since $\mathbf{F}$ is vertical, we only need the last component of $\mathrm{d}\mathbf{S}$:

$$\mathrm{d}\mathbf{S} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(t) & \sin(t) & ? \\ -r\sin(t) & r\cos(t) & ? \end{vmatrix} \mathrm{d}r\mathrm{d}t = (?\mathbf{i} + ?\mathbf{j} + r\mathbf{k})\mathrm{d}r\mathrm{d}t.$$

Using that $\mathbf{F} = a^2\mathbf{k}$ on $\mathcal{S}_1$, we obtain:

$$\iint_{\mathcal{S}_1} \mathbf{F}\cdot \mathrm{d}\mathbf{S} = \int_0^{2\pi} \mathrm{d}t \int_0^a a^2 r\mathrm{d}r = \pi a^4.$$

d): We use spherical coordinates on D:

$$x = R\sin(\phi)\cos(\theta),\ y = R\sin(\phi)\sin(\theta),\ z = R\cos(\phi).$$

Then

$$\iiint_D z\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^{2\pi}\mathrm{d}\theta \int_0^{\pi/2}\mathrm{d}\phi \int_0^a (R\cos(\phi))R^2\sin(\phi)\mathrm{d}R =$$

$$= 2\pi\frac{a^4}{4}\int_0^{\pi/2}\cos(\phi)\sin(\phi)\mathrm{d}\phi = \pi\frac{a^4}{2}\left[\frac{\sin^2(\phi)}{2}\right]_{\phi=0}^{\phi=\pi/2} = \pi\frac{a^4}{4}.$$

e): Note that $\mathrm{div}(\mathbf{F}) = 2z$. By the Divergence Theorem, c) and d), we have:

$$\iint_{\mathcal{S}_2} \mathbf{F}\cdot\mathrm{d}\mathbf{S} = \iiint_D \mathrm{div}(\mathbf{F})\mathrm{d}x\mathrm{d}y\mathrm{d}z - \iint_{\mathcal{S}_1}\mathbf{F}\cdot\mathrm{d}\mathbf{S} =$$

$$= 2\iiint_D z\mathrm{d}x\mathrm{d}y\mathrm{d}z - \pi a^4 = \frac{\pi a^4}{2} - \pi a^4 = -\frac{\pi a^4}{2}$$

Let us check that this result is indeed correct. In the parametrization $(x,y) \mapsto (x,y,0)$ of $\mathcal{S}_2$, note that $\mathrm{d}\mathbf{S} = -\mathbf{k}\mathrm{d}x\mathrm{d}y$, and $F = (x^2+y^2)\mathbf{k}$. Therefore

$$\iint_{\mathcal{S}_2}\mathbf{F}\cdot\mathrm{d}\mathbf{S} = \iint_{x^2+y^2\le a^2} -(x^2+y^2)\mathrm{d}x\mathrm{d}y = -\int_0^{2\pi}\mathrm{d}\theta\int_0^a r^2 r\mathrm{d}r = -2\pi a^4/4 = -\frac{\pi a^4}{2}.$$

□