Solution to 1. We calculate the volume by iteration:

$$\iiint_R \mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^1 \mathrm{d}y \int_{1-\sin(y^2)}^{e^{y^2}} \mathrm{d}z \int_0^y \mathrm{d}x =$$
$$= \int_0^1 y(e^{y^2} - 1 + \sin(y^2))\ \mathrm{d}y =$$
$$= \frac{1}{2}\left(e^{y^2} - y^2 - \cos(y^2)\right)\Big|_{y=0}^{y=1} =$$
$$= \frac{1}{2}\left(e - 1 - \cos(1)\right).$$

□

Solution to 2. *Part a)*. We have that:

$$\oint_{\mathcal{C}} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_0^{2\pi} ((p\cos(\theta) + q\sin(\theta))\mathbf{i} + (r\cos(\theta) + s\sin(\theta))\mathbf{j})\cdot(-\sin(\theta)\mathbf{i} + \cos(\theta)\mathbf{j})\,\mathrm{d}\theta =$$
$$= \int_0^{2\pi} \left(-p\cos(\theta)\sin(\theta) - q\sin^2(\theta) + r\cos^2(\theta) + s\sin(\theta)\cos(\theta)\right)\mathrm{d}\theta =$$
$$= \pi(r-q),$$

where we have used that

$$\int_0^{2\pi} \sin(\theta)\cos(\theta) = 0, \qquad \int_0^{2\pi} \sin^2(\theta) = \int_0^{2\pi}\cos^2(\theta) = \pi.$$

Part b). We have that

$$\mathrm{curl}(\mathbf{F}) = \left(\frac{\partial}{\partial x}\left(\frac{rx+sy}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{px+qy}{x^2+y^2}\right)\right)\mathbf{k} =$$
$$= \frac{-(q+r)(x^2-y^2) + 2xy(p-s)}{(x^2+y^2)^2}\mathbf{k}.$$

Thus, we have $\mathrm{curl}(\mathbf{F}) = 0$ if and only if $s = p$ and $r = -q$; and this gives the answer to *b)*.

Part c). Assume now that $\mathbf{F}$ is conservative. Since conservative vector fields are irrotational, by *b)* it follows that $s = p$ and $r = -q$. Since $\mathbf{F}$ is conservative its circulation over every closed curve is zero; thus, by *a)*, $0 = \pi(r-q)$; and therefore $r = q = 0$. Thus, if $\mathbf{F}$ is conservative, it must be of the form:

$$\mathbf{F}(x,y) = p\left(\frac{x}{x^2+y^2}\mathbf{i} + \frac{y}{x^2+y^2}\mathbf{j}\right).$$

Conversely, vector fields of this form are conservative, with potential function:

$$f : \mathbb{R}^2\backslash\{(0,0)\} \to \mathbb{R}, \qquad f(x,y) = p\,\ln(\sqrt{x^2+y^2}) = p/2\,\ln(x^2+y^2).$$

This completes the proof of *c)*.

□

Solution to 3. We have that div $\mathbf{F} = 3$. In polar coordinates $x = r\cos(\theta)$, $y = r\sin(\theta)$ and $z = z$, we have that $\mathrm{d}x\mathrm{d}y\mathrm{d}z = r\mathrm{d}\theta\mathrm{d}r\mathrm{d}z$, and D is described by:

$$0 \le \theta \le 2\pi, \quad -a \le z \le a, \quad a \le r \le \sqrt{2a^2 - z^2}.$$

Therefore:

$$\iiint_D \mathrm{div}\ \mathbf{F}\ \mathrm{d}x\mathrm{d}y\mathrm{d}z = 3\int_0^{2\pi}\mathrm{d}\theta \int_{-a}^{a}\mathrm{d}z \int_a^{\sqrt{2a^2-z^2}} r\ \mathrm{d}r =$$
$$= 6\pi\int_{-a}^{a}\frac{1}{2}\left(a^2 - z^2\right)\mathrm{d}z =$$
$$= 3\pi\left(2a^3 - \frac{2}{3}a^3\right) =$$
$$= 4\pi a^3.$$

Let $\mathcal{S}_1$ be the cylindric part of the boundary of D, i.e. in cylindrical coordinates $\mathcal{S}_1$ is described by:

$$0 \le \theta \le 2\pi, \quad r = a, \quad -a \le z \le a.$$

Orienting $\mathcal{S}_1$ such that the positive side is the outside of D, the standard formulas for the area element give:

$$\mathrm{d}\mathbf{S} = -a(\cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j})\ \mathrm{d}\theta\mathrm{d}z.$$

Therefore,

$$\iint_{\mathcal{S}_1} \mathbf{F}\cdot\mathrm{d}\mathbf{S} = -a^2\int_0^{2\pi}\mathrm{d}\theta\int_{-a}^{a}\mathrm{d}z = -4\pi a^3.$$

Let $\mathcal{S}_2$ be the spherical part of the boundary of D:

$$x^2+y^2+z^2 = 2a^2,\quad a^2 \le x^2+y^2.$$

Using spherical coordinates

$$x = R\sin(\phi)\cos(\theta),\quad y = R\sin(\phi)\sin(\theta),\quad z = R\cos(\phi),$$

$\mathcal{S}_2$ is described by:

$$0\le\theta\le 2\pi,\quad R=\sqrt{2}a,\quad \pi/4\le\phi\le 3\pi/4.$$

Orienting $\mathcal{S}_2$ such that the positive side is the outside of D, the standard formulas for the area element give:

$$\mathrm{d}\mathbf{S} = 2a^2\sin(\phi)\left(\sin(\phi)(\cos(\theta)\mathbf{i}+\sin(\theta)\mathbf{j})+\cos(\phi)\mathbf{k}\right)\ \mathrm{d}\theta\mathrm{d}\phi = 2a^2\sin(\phi)\mathbf{N}\mathrm{d}\theta\mathrm{d}\phi,$$

where $\mathbf{N}$ is the unit normal vector to the sphere pointing outwards. Since along $\mathcal{S}_2$ we have that $\mathbf{F} = \sqrt{2}a\mathbf{N}$, we obtain:

$$\begin{aligned}\iint_{\mathcal{S}_2}\mathbf{F}\cdot\mathrm{d}\mathbf{S} &= 2\sqrt{2}a^3\int_0^{2\pi}\mathrm{d}\theta\int_{\pi/4}^{3\pi/4}\sin(\phi)\mathrm{d}\phi = \\ &= 4\sqrt{2}\pi a^3\left(\cos(\pi/4)-\cos(3\pi/4)\right) = \\ &= 8\pi a^3.\end{aligned}$$

We have calculated both sides of the divergence theorem for $\mathbf{F}$ on D:

$$\iiint_D \operatorname{div}\mathbf{F}\ \mathrm{d}x\mathrm{d}y\mathrm{d}z = 4\pi a^3 = (-4\pi a^3)+(8\pi a^3) = \iint_{\mathcal{S}_1}\mathbf{F}\cdot\mathrm{d}\mathbf{S} + \iint_{\mathcal{S}_2}\mathbf{F}\cdot\mathrm{d}\mathbf{S}.$$ □

Solution to 4. Part b) We have that

$$\operatorname{curl}(\mathbf{F}) = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y^2+y-1 & -xyz & 0\end{vmatrix} = xy\mathbf{i} - (y(z+2)+1)\mathbf{k}.$$

Part c). Using polar coordinates in the plane, and regarding $\mathcal{S}$ as a graph, we obtain the following parametrization of $\mathcal{S}$:

$$x(\theta,r) = r\cos(\theta),\quad y(\theta,r) = r\sin(\theta),\quad z(\theta,r) = r,$$
$$0\le\theta\le 2\pi,\quad 1\le r\le 2.$$

In this parameterization, we have that

$$\mathrm{d}\mathbf{S} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -r\sin(\theta) & r\cos(\theta) & 0\\ \cos(\theta) & \sin(\theta) & 1\end{vmatrix}\mathrm{d}\theta\mathrm{d}r = (r\cos(\theta)\mathbf{i}+r\sin(\theta)\mathbf{j}-r\mathbf{k})\,\mathrm{d}\theta\mathrm{d}r$$

Note that $\mathrm{d}\mathbf{S}$ points in the positive side of the surface. Thus, we have:

$$\begin{aligned}\operatorname{curl}(\mathbf{F})\cdot\mathrm{d}\mathbf{S} &= \left(r^2\cos(\theta)\sin(\theta)\mathbf{i} - (r(r+2)\sin(\theta)+1)\mathbf{k}\right)(r\cos(\theta)\mathbf{i}+r\sin(\theta)\mathbf{j}-r\mathbf{k})\,\mathrm{d}\theta\mathrm{d}r = \\ &= \left(r^3\cos^2(\theta)\sin(\theta)+r^2(r+2)\sin(\theta)+r\right)\mathrm{d}\theta\mathrm{d}r.\end{aligned}$$

Finally, we compute the flux:

$$\iint_{\mathcal{S}}\operatorname{curl}(\mathbf{F})\cdot\mathrm{d}\mathbf{S} = \int_0^{2\pi}\int_1^2\left(r^3\cos^2(\theta)\sin(\theta)+r^2(r+2)\sin(\theta)+r\right)\mathrm{d}\theta\mathrm{d}r = 3\pi.$$

Part d). Using the following positive parameterization of $\mathcal{C}_1$:

$$\mathbf{r}(\theta) = \cos(\theta)\mathbf{i}+\sin(\theta)\mathbf{j}+\mathbf{k},$$

we obtain that the circulation is given by:

$$\oint_{\mathcal{C}_1} (y^2 + y - 1)\mathrm{d}x - xyz\mathrm{d}y =$$

$$= \int_0^{2\pi} \left((\sin^2(\theta) + \sin(\theta) - 1)(-\sin(\theta)) - \cos(\theta)\sin(\theta)\cos(\theta)\right)\mathrm{d}\theta =$$

$$= \int_0^{2\pi} \sin(\theta)\left(-\sin(\theta) + 1 - \sin^2(\theta) - \cos^2(\theta)\right)\mathrm{d}\theta =$$

$$= -\int_0^{2\pi} \sin^2(\theta)\mathrm{d}\theta = -\pi.$$

Part e). By Stokes' Theorem, we have that

$$\oint_{\mathcal{C}_1} \mathbf{F}\cdot\mathrm{d}\mathbf{r} + \oint_{\mathcal{C}_2} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \iint_{\mathcal{S}} \mathrm{curl}(\mathbf{F})\cdot\mathrm{d}\mathbf{S}.$$

Therefore, by *c*) and *d*), we have:

$$\oint_{\mathcal{C}_2} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = 4\pi.$$

□