SOLUTIONS TO CALCULUS 4 EXAM
AUGUST 6, 2015

Problem 1

The inequalities describing R are equivalent to:

$$0 \le y \le 1, \quad 0 \le x \ \le e^{y^2}, \quad y/2 \le z \le y.$$

We calculate the volume by iteration:

$$\iiint_R \mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^1 \mathrm{d}y \int_0^{e^{y^2}} \mathrm{d}x \int_{y/2}^{y} \mathrm{d}z =$$

$$= \int_0^1 e^{y^2}\frac{y}{2}\mathrm{d}y = \frac{1}{4}e^{y^2}\Big|_{y=0}^{y=1} = \frac{e-1}{4}.$$

Problem 2

a) If (u,v) is in A, then $u^2+v^2 \le 1$. Thus:

$$(1-x(u,v))x^3(u,v) - y^2(u,v) = (1-u^2)u^6 - u^6v^2 = u^6(1-u^2-v^2) \ge 0.$$

Hence $(x(u,v), y(u,v))$ is in B.

b) The Jacobian determinant of the transformation is:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x(u,v)}{\partial u} & \frac{\partial x(u,v)}{\partial v} \\ \frac{\partial y(u,v)}{\partial u} & \frac{\partial y(u,v)}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & 0 \\ 3u^2v & u^3 \end{vmatrix} = 2u^4.$$

Therefore, the area element becomes:

$$\mathrm{d}x\mathrm{d}y = \frac{\partial(x,y)}{\partial(u,v)}\mathrm{d}u\mathrm{d}v = 2u^4\mathrm{d}u\mathrm{d}v.$$

c) We perform the change of coordinates $x = x(u,v)$, $y = y(u,v)$:

$$\iint_B \frac{x^4+y^2}{x^5}\mathrm{d}x\mathrm{d}y = \iint_A \frac{u^8+u^6v^2}{u^{10}}2u^4\mathrm{d}u\mathrm{d}v = \iint_A 2(u^2+v^2)\mathrm{d}u\mathrm{d}v.$$

Finally, we pass to polar coordinates $u = r\cos(\theta)$, $v = r\sin(\theta)$:

$$\iint_A 2(u^2+v^2)\mathrm{d}u\mathrm{d}v = 2\int_{-\pi/2}^{\pi/2}\mathrm{d}\theta \int_0^1 r^3\mathrm{d}r = 2\pi\frac{1}{4} = \frac{\pi}{2}.$$

Problem 3

b) We decompose the surface $\mathcal{S}$ of D into 5 parts: $\mathcal{S}_1$, ..., $\mathcal{S}_5$. On each part $\mathcal{S}_i$ we calculate: the normal vector $\mathbf{N}_i$ of $\mathcal{S}_i$ pointing out of D, the value of $\mathbf{F}\cdot\mathbf{N}_i$ on $\mathcal{S}_i$, and then flux of $\mathbf{F}$ across $\mathcal{S}_i$.

$$\mathcal{S}_1: \ x=0,\ 0\le y,z,y+z\le 1,\quad \mathbf{N}_1=-\mathbf{i},$$

$$\mathbf{F}\cdot\mathbf{N}_1=(yz\mathbf{j}-yz\mathbf{k})\cdot(-\mathbf{i})=0,\quad \iint_{\mathcal{S}_1}\mathbf{F}\cdot\mathbf{N}_1\mathrm{d}S=0;$$

$$\mathcal{S}_2: \ y=0,\ 0\le x,z\le 1,\quad \mathbf{N}_2=-\mathbf{j},$$

$$\mathbf{F}\cdot\mathbf{N}_2=(e^z x\mathbf{i})\cdot(-\mathbf{j})=0,\quad \iint_{\mathcal{S}_2}\mathbf{F}\cdot\mathbf{N}_2\mathrm{d}S=0;$$

$$\mathcal{S}_3: \ z=0,\ 0\le x,y\le 1,\quad \mathbf{N}_3=-\mathbf{k},$$

$$\mathbf{F}\cdot\mathbf{N}_3=(x\mathbf{i})\cdot(-\mathbf{k})=0,\quad \iint_{\mathcal{S}_3}\mathbf{F}\cdot\mathbf{N}_3\mathrm{d}S=0;$$

$$\mathcal{S}_4: \ x=1,\ 0\le y,z,y+z\le 1,\quad \mathbf{N}_4=\mathbf{i},$$

$$\mathbf{F}\cdot\mathbf{N}_4=(e^z\mathbf{i}+yz\mathbf{j}-yz\mathbf{k})\cdot\mathbf{i}=e^z,$$

$$\iint_{\mathcal{S}_4}\mathbf{F}\cdot\mathbf{N}_4\mathrm{d}S=\int_0^1\mathrm{d}y\int_0^{1-y}e^z\mathrm{d}z=\int_0^1(e^{1-y}-1)\mathrm{d}y=e-2;$$

$$\mathcal{S}_5: \ 0\le x,y,z\le 1,\ y+z=1,\quad \mathbf{N}_5=\frac{1}{\sqrt{2}}(\mathbf{j}+\mathbf{k}),$$

$$\mathbf{F}\cdot\mathbf{N}_5=\frac{1}{\sqrt{2}}(yz-yz)=0,\quad \iint_{\mathcal{S}_5}\mathbf{F}\cdot\mathbf{N}_5\mathrm{d}S=0.$$

Thus, we obtain that the flux of $\mathbf{F}$ out of $\mathcal{S}$ is given by:

$$\iint_{\mathcal{S}}\mathbf{F}\cdot\mathbf{N}\mathrm{d}S=e-2.$$

We have that $\mathbf{div}\mathbf{F}=e^z+y-z$. So:

$$\iiint_D \mathbf{div}\mathbf{F}\mathrm{d}x\mathrm{d}y\mathrm{d}z=\int_0^1\mathrm{d}x\int_0^1\mathrm{d}y\int_0^{1-y}(e^z+z-y)\mathrm{d}z=$$

$$=\int_0^1\left(e^{1-y}-1+\frac{1}{2}(1-y)^2-y(1-y)\right)\mathrm{d}y=e-2-\frac{1}{6}(0-1)-\frac{1}{2}+\frac{1}{3}=e-2.$$

So, we have checked that the divergence theorem for $\mathbf{F}$ on D holds:

$$\iint_{\mathcal{S}}\mathbf{F}\cdot\mathbf{N}\mathrm{d}S=e-2=\iiint_D\mathbf{div}\mathbf{F}\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

Problem 4

b) We use the following parameterization of the curve $\mathcal{C}$:

$$\mathbf{r}(\theta) = (\cos(\theta), \sin(\theta), \cos^2(\theta) - \sin^2(\theta)) = (\cos(\theta), \sin(\theta), \cos(2\theta)),$$

$0 \le \theta \le 2\pi$. Then

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\theta}(\theta) = (-\sin(\theta), \cos(\theta), -2\sin(2\theta)).$$

The length element becomes:

$$\mathrm{d}s = \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\theta}\right| = \sqrt{\sin^2(\theta) + \cos^2(\theta) + 4\sin^2(2\theta)} =$$
$$= \sqrt{1 + 4(1 - \cos^2(2\theta))} = \sqrt{5 - 4\cos^2(2\theta)}.$$

Since in our parameterization, $z(\theta) = \cos(2\theta)$, the line integral has the following value:

$$\int_{\mathcal{C}} \frac{1}{\sqrt{5 - 4z^2}} \mathrm{d}s = \int_0^{2\pi} \mathrm{d}\theta = 2\pi.$$

c) We use the standard formula for calculating $\mathrm{d}\mathbf{S}$:

$$\mathrm{d}\mathbf{S} = \frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} \mathrm{d}x\mathrm{d}y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2x \\ 0 & 1 & -2y \end{vmatrix} \mathrm{d}x\mathrm{d}y = (-2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k})\,\mathrm{d}x\mathrm{d}y.$$

d) We have:

$$\mathbf{grad}(g) = y\mathbf{i} + (x + \cos(y)e^{2z})\mathbf{j} + 2\sin(y)e^{2z}\mathbf{k}.$$

e)*Solution 1.* Note that, by *d*), the following relation holds:

$$\oint_{\mathcal{C}} (2x + \cos(y)e^{2z})\mathrm{d}y + 2\sin(y)e^{2z}\mathrm{d}z = \oint_{\mathcal{C}} \mathbf{grad}(g) \cdot \mathrm{d}\mathbf{s} + \oint_{C} -y\mathrm{d}x + x\mathrm{d}y.$$

The first term on the right-hand-side is zero, because the integral of the gradient of any function on a closed curve is zero. Using the parameterization from *b*), we obtain that the total line integral equals:

$$\oint_{C} -y\mathrm{d}x + x\mathrm{d}y = \int_0^{2\pi} \left(\sin^2(\theta) + \cos^2(\theta)\right) \mathrm{d}\theta = 2\pi.$$

e)*Solution 2.* We use Stokes' theorem for the vector field:

$$\mathbf{G} = (2x + \cos(y)e^{2z})\mathbf{j} + 2\sin(y)e^{2z}\mathbf{k}.$$

Again, using *d*), we have that:

$$\mathbf{G} = \mathbf{grad}(g) + \mathbf{F}, \quad \mathbf{F} = -y\mathbf{i} + x\mathbf{j},$$

and since $\mathbf{curl}(\mathbf{grad}(g)) = 0$, it follows that:

$$\mathbf{curl}(\mathbf{G}) = \mathbf{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = 2\mathbf{k}.$$

Of course, the equality $\mathbf{curl}(\mathbf{G}) = 2\mathbf{k}$ can be obtained also by a direct calculation. So, by Stokes' Theorem, and c) we have that:

$$\oint_{\mathcal{C}} \mathbf{G} \cdot \mathrm{d}\mathbf{s} = \iint_{\mathcal{S}} \mathbf{curl}(\mathbf{G}) \cdot \mathrm{d}\mathbf{S} = \iint_{x^2+y^2\leq 1} 2\mathrm{d}x\mathrm{d}y = 2\pi.$$