Questions Resit Exam NWI-WB046B (only written part)

1. (10 points) **Maximum principle**

   Assume $u \in C^{1,2}(\Omega_T)$ with $\Omega_T = (0,T] \times \Omega$ for $\Omega \subseteq \mathbb{R}^n$ bounded and open. Consider the partial differential operator

   $$Lu := u_t - \Delta u + \sum_{i=1}^n b_i u_{x_i} + cu, \qquad \text{on } \Omega_T,$$

   with $b = (b_1, \dots, b_n) \in \mathbb{R}^n$ and $c \in C(\overline{\Omega_T})$. Show that if $Lu \leq 0$ and $c \geq 0$ on Ω_T then

   $$\max_{\overline{\Omega_T}} u \leq \max_{\Gamma_T} u^+,$$

   where $u^+ = \max(u, 0)$ is the positive part of u and $\Gamma_T := ([0,T] \times \partial\Omega) \cup (\{0\} \times \Omega)$.

   (Hint: Distinguish several cases. For the full case $Lu \leq 0$ use a trial function u_ε, for example, $u_\varepsilon = u + \varepsilon \exp(\alpha x_1)$ for a suitable choice of α.)

2. (10 points) **Decaying solutions**

   Let u solve the equation

   $$\begin{aligned} u_{tt} - \Delta u &= 0, && \text{in } [0,\infty) \times \mathbb{R}^3, \\ u &= u_0, && \text{on } \{t = 0\} \times \mathbb{R}^3, \\ u_t &= u_1, && \text{on } \{t = 0\} \times \mathbb{R}^3, \end{aligned}$$

   where u_0 and u_1 are smooth and have compact support. Show there exists a constant c such that

   $$|u(t,x)| \leq \frac{c}{t}, \qquad \text{for all } x \in \mathbb{R}^3, t > 0.$$

3. (10 points) **Method of characteristics**

   (a) Solve the following initial value problem:

   $$\begin{aligned} 3u_{x_1} - 3u_{x_2} &= u^2 && \text{in } (0,\infty) \times \mathbb{R}, \\ u &= g && \text{on } \Gamma = \{x_2 = 0\}. \end{aligned}$$

   Here, g is a given function. What regularity of g is required to obtain a classical solution?

   (b) Let $\Omega \subseteq \mathbb{R}^n$ be open with nonempty C^1 boundary $\partial\Omega$. Consider a first-order partial differential equations with given boundary data of the form

   $$\begin{aligned} F(\nabla u, u, x) &= 0 && \text{in } \Omega \subseteq \mathbb{R}^n, \\ u &= g && \text{on } \Gamma \subseteq \partial\Omega. \end{aligned}$$

   Suppose $\Gamma \subseteq \{x_n = 0\}$ locally, F is C^2 and g is C^1. Assume furthermore that (p^0, z^0, x^0) is an admissible triple (of initial data) for the system of characteristic equations for (p, z, x). Answer the following questions briefly:

   (i) What additional assumption is required to obtain a unique local solution for the system of characteristic equations? (name and condition needed)

   (ii) Why is this assumption required for the proof of local invertibility? (one sentence is enough, *no* proof is needed!)

   (iii) What does it mean intuitively/geometrically?