Questions Resit Exam NWI-WB107 (only second part)

1. (10 points) **Euler–Lagrange equation**

   (a) Compute all solutions to the Euler–Lagrange equation related to the functional

   $$J(y) = \int_0^1 (y')^2 + y\,dx$$

   in $C^{1,\text{pw}}[0,1]$.

   (b) Which solutions satisfy the natural boundary conditions?

   (c) Which solutions satisfy the boundary conditions $y(0) = 0$, $y(1) = 1$?

   (d) Which solutions are local extrema with/without boundary conditions? Are they global extrema?

2. (10 points) **Shortest curves on a cylinder**

   Let us consider a cylinder in $\mathbb{R}^3$, given by Cartesian coordinates (x, y, z) such that $x^2+y^2 = R^2$ (R is the radius, and constant) and $z \in \mathbb{R}$ is the height. Consider two points P_1 and P_2 on the cylinder, a curve γ between them and the arc-length functional

   $$L(\gamma) = \int_{P_1}^{P_2} dS = \int_{P_1}^{P_2} \sqrt{dx^2 + dy^2 + dz^2}.$$

   Answer the following questions.

   (a) Express the Cartesian coordinates (x, y) in terms of polar coordinates (R is fixed, θ variable angle) and show that the arc-length is then

   $$L(z) = \int_{\theta_1}^{\theta_2} \sqrt{R^2 + \left(\frac{dz}{d\theta}\right)^2}\,d\theta,$$

   for a curve given by $\theta \mapsto z(\theta)$.

   (b) Derive a differential equation for the shortest curves on the cylinder (assuming sufficient regularity).

   (c) What geometric shape do the curves in (b) have? Solve the differential equation and compute an explicit solution for $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2)$ with $(x_1, y_1) \neq (x_2, y_2)$.

   (d) Are the solutions obtained in (b) indeed length-minimizing? Why (not)?